CONTROL OF SWITCHED LINEAR SYSTEMS
ADAPTATION AND ROBUSTNESS

Shuai Yuan

CONTROL OF SWITCHED LINEAR SYSTEMS
ADAPTATION AND ROBUSTNESS

Proefschrift

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. dr. ir. T.H.J.J. van der Hagen,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op 5 juli 2018 om 10.00 uur
door

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Dit proefschrift is goedgekeurd door de:

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Samenstelling van de doctoraatscommissie:

Rector Magnificus	voorzitter
Prof. dr. ir. B. De Schutter	Technische Universiteit Delft, promotor
Dr. ir. S. Baldi	Technische Universiteit Delft, copromotor

Onafhankelijke leden:

Prof.dr. L. Zhang	Harbin Institute of Technology
Prof.dr. P. Colaneri	Polytechnic University of Milan
Prof.dr.ir. W.P.M.H. Heemels	Technische Universiteit Eindhoven
Prof.dr.ing. H. Vallery	Technische Universiteit Delft
Prof.dr.ir. J.W. van Wingerden	Technische Universiteit Delft

Research described in this thesis was supported by the China Scholarship Council (CSC) under grant 20146160098, the Marie-Curie action FP7-PEOPLE-451-2012-IAPP 'Advanced Methods for Building Diagnostics and Maintenance' (AMBI), and the Delft Center for Systems and Control.

Published and distributed by: Shuai Yuan
E-mail: xiaoshuaihust@hotmail.com

ISBN 978-94-6186-937-1

Keywords: switched linear systems, parametric uncertainties, adaptive control, robust control, time delays

Printed in the Netherlands

Acknowledgements

Four years ago, I set foot in this beautiful and unfamiliar country to start my PhD journey. Just like the feeling for the differences between eastern and western culture, I was excited and nervous about the numerous challenges and uncertainties of my future academic life. Although I have experienced depression and anxiety, the journey turns out to be enjoyable and exciting with the help from many people. Here, I want to express my sincere gratitude to them.

Foremost, I would like to thank my daily supervisor dr. Simone Baldi for his comprehensive and professional supervision. Since I had no background on related fields when starting my journey, Simone imparted me a lot of basic knowledge in the early stages with great profession and patience. Whenever I had technical questions, he was always in the position to give insightful suggestions about how to solve them rather than the solutions. In this way, I get to know how to do research independently step by step. Apart from research, Simone also helped me to interact with industry and I got the chance to work at Honeywell Laboratory Prague for two months. My sincere thanks also go to my promoter prof. Bart De Schutter for his critical and professional supervision of my PhD research and for continuous support whenever I had difficult times. Besides, I would like to express my wholehearted gratitude to prof. Lixian Zhang from Harbin Institute of Technology for the valuable discussions and suggestions of related work that illuminate the way to future research.

Second, I would like to thank my PhD committee members, prof. Patrizio Colaneri, prof. Maurice Heemels, prof. Heike Vallery, and prof. Jan-Willem van Wingerden for their valuable comments from various perspectives to improve my thesis.

I want to thank my good friends: Xiao Lin for helping me adapt to a new life in the Netherlands four years ago; Matiya and Tina for lots of wonderful moments we had in Delft and when travelling together in Slovenia and China as well; my flatmates Hai Gong, Pengling Wang, and Zhi Hong for preparing dinner and watching movies together in the last three years, which created lots of happy and relaxing times; Guohui, Mi, Kehuan, and Junquan for their company in London during my visiting study in Imperial College. I am grateful to Zhenji, Minghe, Fanyu, Bo, Wei, Yibing, Jiapeng, Yuan, Xiang, Jian, Xin, Yue, Jingtao, jiakun, xinyuan, Meng, Jiao, Fei, Yaming, Xiaodong, Huatang, Meixia, Xing, Qu, Wen for their sincere friendship.

I am very happy that I have so many lovely colleagues at DCSC. I would like to thank Chengpu and his family, Yiming, Fan, and Zhe for sharing their research experiences and understandings. I thank my officemates Farid and Filippo for their company and for sharing happy moments and Vahab for helping me get to know new study environment in the early stages. Many thanks go to Anqi, Arman, Anahita, Cees, Dean, Dieky, Eunice, Edwin, Elisabeth, Hai, Huizhen, Hans, Jeroen, Jia, Jun, Le, Laura, Laurens, Maolong, Max, Mohammad, Na, Nikolaos, Pieter, Reinier, Renshi, Shuai Liu, Su, Tomas, Yuzhang, Yu, Yue, Yihui, et al.

for joyous moments of table football (though I still need to improve my skills), social events, dining, etc. Thanks Pieter Piscaer for translating my dissertation summary into Dutch. I also thank the secretaries of DCSC: Kitty, Marieke, Hellen, and Kiran for their kindness and assistance.

Next, I am deeply grateful to my master's supervisor prof. Fangyu Peng from Huazhong University of Science and Technology for his suggestions and support when I decided to study abroad. Without his help, I might not have the chance to have a beautiful time in the Netherlands and my life trajectory would evolve towards a totally different direction.

I would like to use this opportunity to thank my girlfriend Lvyin Cai for her love, support, and understanding. Although being thousands miles away, she was always there to encourage me when I had difficult moments, to cheer me up when I felt depressed, and to offer me suggestions when I encountered problems. Thank you so much for standing behind me during my PhD study. Last, but by no means least, I would like to express my deepest gratitude to my beloved parents for their tremendous support and unreserved love.

Shuai Yuan

Delft, June 2018

Notations

$\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$	sets of real numbers, n component real vectors, and n by n real matrices
$\mathbb{N}$, $\mathbb{N}^+$	set of natural numbers and positive natural numbers
x^T, X^T	transpose of vector x, matrix X
$P = P^T > 0$	positive definite symmetric matrix
$\mathrm{tr}(\cdot)$	trace of a matrix
I_n	identity matrix of dimension n
$\\|\cdot\\|$	Euclidean norm
$\square$	end of proof or remark
$:=$	equal by definition
$\mathrm{sgn}(\cdot)$	sign of a number
$\mathrm{diag}\{\cdots\}$	a block-diagonal matrix
$\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$	minimum and maximum eigenvalues of a square matrix
$\varphi(t^-)$	left limit of $\varphi(t)$, i.e., $\varphi(t^-) = \lim_{\tau\to t^-}\varphi(\tau)$
$\mathscr{L}_2, \mathscr{L}_\infty$	space of square-integrable, bounded functions
$\mathscr{L}_2^r$	set of square integrable functions with values on $\mathbb{R}^r$ defined on $[0,\infty)$
Class $\mathcal{K}$	a function $\alpha : [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$
Class $\mathcal{KL}$	a function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$ if it is of *class* $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$
Class $\mathcal{K}_\infty$	a function γ: $[0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}_\infty$ if it is continuous strictly increasing, unbounded, and $\gamma(0) = 0$

Contents

Chapter 1

Introduction

In this chapter, we first present the motivation of the research of this thesis according to the following two main research directions: adaptive stabilization of switched linear systems with parametric uncertainties; adaptive and robust stabilization of switched linear systems with time delays. The research questions and main contribution of this work are given. After that, the chapter is concluded with a brief outline of the thesis.

1.1 Motivation of the research

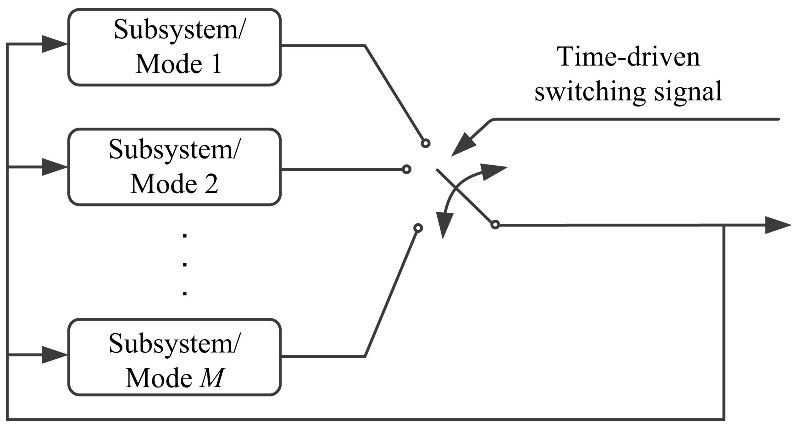

Figure 1.1: The framework of time-driven switched systems.

Switched systems are a special class of hybrid systems that consists of collections of subsystems (or modes) with continuous dynamics, and a rule to regulate the switching behavior between them, called *switching law or signal.* Based on the nature of the switching signals, switched linear systems can be generally categorized into two classes: state-dependent switched systems and time-driven switched systems. In this thesis, we focus on switched systems with time-driven switching signals, cf. Fig. 1.1, due to their applications to a broad range of complex physical systems whose dynamics changes from time to time, such as networked control systems [28], automotive systems [62], smart energy systems [139], fluid mixing [27], flight control systems [91]. To be specific, the application of time-driven switched systems to a flight control system is elaborated [91]: an aircraft system tends to display different dynamics at different operating points specified by the vehicle speed and altitude; the switch of operating point can not be activated too often in order to avoid instability. When controlling such complex systems, parameter uncertainties and external disturbances are regarded as crucial undesired factors that should be addressed. This creates additional difficulties when designing control and switching laws. In general, there are two main families of techniques dealing with controller design of systems with parameter uncertainties and disturbances: robust control and adaptive control. Robust control methods with fixed control parameters are used to guarantee system stability given that the parametric uncertainties or disturbances are confined within a known compact set, while adaptive control methods with adjustable control parameters can be adopted to deal with uncertainties and disturbances within a possibly unknown compact set. In addition, time delay is another crucial factor of complex systems that needs investigation. In view of this, this thesis is organized in two parts: adaptive control of switched linear systems with parameter uncertainties and disturbances; robust and adaptive stabilization of switched linear systems with time delays.

1.1.1 Adaptive stabilization of switched linear systems

To date, productive research has been conducted on the fundamental problems of stability and stabilization of time-driven switched systems [9, 21, 33, 37, 38, 63, 77, 78, 94, 140]. Two families of switching laws have been mainly considered: dwell time and average dwell time. For dwell time switching, it is imposed that the switching interval between two consecutive discontinuities of the switching signal is larger than a sufficiently large constant. For average dwell time switching, the switching interval between two consecutive discontinuities of the switching signal is sufficiently large in an average sense: this means that very short switching intervals are allowed provided that they are compensated by long ones. It is clear that average dwell time switching relaxes the concept of dwell time switching. Subsequently, conservativeness[1] of average dwell time switching has been further decreased by a new switching strategy proposed in [154]: mode-dependent average dwell time switching. The peculiarity of this switching strategy consists in exploiting the information of every mode, such as the exponential rate of the Lyapunov function associated to each mode.

On the other hand, being built upon the basis of the stability results for switched systems, research on uncertain switched systems using adaptive techniques is not equally mature. It is well recognized that a single robust controller may lead to very conservative performance for a large uncertainty set [60, 79, 96]. Therefore, when the uncertainties are polytopic, using a family of robust controllers has been proposed to improve the performance of a single controller [1]. As a complement to robust control, adaptive approaches for non-switched uncertain systems have been investigated to improve the performance of robust approaches over large non-polytopic uncertainties [5, 51, 75, 103]. However, adaptive control of uncertain switched systems is more challenging than robust control. This is because adaptive closed-loop systems are intrinsically nonlinear: not only an adaptive law should be developed to estimate the unknown parameters, but also a switching law should be carefully designed to guarantee the stability of the closed-loop system. Recently, some research has been conducted on adaptive control of uncertain switched linear systems, i.e., switched linear systems with state-dependent switching laws [23–26, 46], and switched linear systems with time-driven switching laws [91, 117, 118]. These two works [25] and [91] can be cited as representative research for uncertain state-dependent switched linear systems and uncertain time-driven switched linear systems, respectively. For state-dependent switched systems,

[1] In this work, the term conservativeness is adopted to indicate the lower bound on the length of the switching intervals for which stability can be guaranteed

di Bernardo et al. [25] developed an adaptive law based on the so-called minimal control synthesis algorithm, which can guarantee that the plant states asymptotically track a reference trajectory. For time-driven switched systems, Sang and Tao [91] proposed a family of adaptive laws with parameter projection and a switching law based on dwell time, which, however, cannot guarantee asymptotic stability of the tracking error if no common Lyapunov function exists. Instead, only a "mean-square" performance of the tracking error during a finite time interval was presented. The problem of mean-square performance is that it does not give information on learning of transient performance, which are important in adaptive control loops. In light of this, the first motivation of this thesis stems from the following question:

Question 1: Can we reduce the conservativeness of the switching laws based on dwell time in [91] and establish bounds on the transient performance and steady-state performance of the tracking error instead of the mean-square performance?

Furthermore, since the well-known results of adaptive control of non-switched linear systems guarantee asymptotic stability of the tracking error and convergence of the parameter estimates to real parameters with the persistent excitation condition [103], one will recognize the existence of a theoretical gap between adaptive control of switched linear systems and adaptive control of non-switched linear systems [91]. In fact, for switched linear systems, asymptotic stability and convergence to the actual parameters have been guaranteed only in the special case of having a common Lyapunov function. Therefore, we target the second fundamental question as follows:

Question 2: Can we develop an adaptive law and a switching law for uncertain switched linear systems to achieve the same asymptotic stability and parameter convergence results as adaptive control of classical non-switched systems?

It is well-established that small disturbances in non-switched systems may lead to instability of the closed-loop systems if robustification techniques, such as parameter projection and leakage, are not employed [44]. In this regard, to preserve stability of switched systems subject to disturbances, modification methods for the adaptive laws of non-switched systems should be extended to the adaptive laws of switched systems. This gives rise to the following question:

Question 3: How to robustify adaptive laws for switched linear systems?

1.1.2 Adaptive and robust stabilization of switched linear systems with time delays

Time-varying delay of the system state is a common problem in switched systems. Time-varying delays cause the state of a system to evolve based on some delayed information [34]. Increasing focus has been given on stability and stabilization of switched systems with time-varying delays [17, 20, 28, 53, 66, 70, 98, 99, 105, 113, 146], where the well-known Krasovskii and Razumikhin techniques for non-switched systems are extended to address time-varying delays of switched systems. However, when switched systems are subject to parameter uncertainties, the aforementioned methods show some limitations when applied to adaptive stabilization of switched time-delay systems. On the one hand, the Krasovskii technique requires derivatives of the time-varying delays to be bounded, i.e., time-varying delay should be continuous at the switching instants [58, 98]. On the other hand, even if the Razumikhin technique does not show the drawback of the Krasovskii technique, its application in an adaptive stabilization setting is not satisfactory: the existence of an adaptive controller cannot be guaranteed [81, 156]. In view of these limitations, a question automatically arises:

Question 4: Can we develop a new technique that can overcome the limitations of the Krasovskii and Razumikhin techniques in the setting of adaptive stabilization of uncertain switched linear systems subject to time-varying delays?

Robust control of switched systems has been also attracting a lot of attention [42, 80, 87, 144, 147, 158]. However, these results mainly focus on an ideal family of switched systems, in which the controller mode switches synchronously with the system mode. In practice, for example, in networked control systems, where the controller communicates with the system through a communication channel, there exist a new type of time delays arising from switching behavior, called switching delays, between the activation of the system mode and the activation of its corresponding controller. This will lead to mismatch between system modes and controller modes. For switched linear systems with switching delays, some results on stability and robust stabilization are presented in [30, 106, 121, 143, 144, 152]. As a fundamental index of robust performance, the $\mathcal{L}_2$ gain of switched systems with switching delays has been studied in [56, 68, 107, 125, 142, 144, 151]: however, for these systems, the classic notion on non-weighted $\mathcal{L}_2$ gain must be relaxed to a weighted $\mathcal{L}_2$ gain with an exponential forgetting factor. This leads to a big inconsistency in the theoretical results of the $\mathcal{L}_2$ gain of non-switched linear systems and of switched linear systems with switching delays. To this end, the following question is proposed:

Question 5: Can we design a robust controller for switched linear systems subject to switching delays that achieves a non-weighted $\mathcal{L}_2$ gain?

1.2 Research goals and main contributions

The research goals of this thesis focus on developing new control schemes to handle uncertainties and disturbances in switched linear systems. The main contribution consists in filling some theoretical gaps between adaptive and robust stabilization of switched linear systems and of non-switched linear systems, which are listed in the following:

- **Adaptive tracking of switched linear systems using extended dwell time and average dwell time**

  We extend the results in [91] using extended notions of dwell time and of average dwell time switching: mode-dependent dwell time and mode-dependent average dwell time switching, respectively. This gives rise to less conservative switching signals. Furthermore, to address the cases in which the next subsystem to be switched to is known, we propose a new time-dependent switching scheme: mode-mode-dependent dwell time switching, which not only exploits the information of the current subsystem, but also of the next subsystem.

- **Adaptive asymptotic tracking of switched linear systems**

  An adaptive law for switched linear systems with parametric uncertainties is introduced, which closes the theoretical gaps between adaptive control of non-switched linear systems and of switched linear systems. The proposed adaptive law and switching law based on dwell time guarantee asymptotic convergence of the tracking error to zero and, with a persistent exciting reference input, convergence of parameter estimates to nominal parameters asymptotically.

- **Robust adaptive tracking of switched linear systems**

  The adaptive law for switched linear systems is modified using the ideas of parameter projection and leakage method, depending on the available *a priori* information: when the bounds of uncertain parameters are known, parameter projection is adopted; otherwise, the leakage method is used. The resulting adaptive closed-loop system is shown to be globally uniformly ultimately bounded in the presence of parametric uncertainties and external disturbances.

- **Adaptive stabilization of switched linear systems with time-varying delays**

  We develop a new adaptive design for uncertain switched linear systems that can address the limitations of the Krasovskii and Razumikhin techniques and handle discontinuities in the time-varying delays. In particular, a stability condition is developed to deal with discontinuities of multiple time-varying delays. By virtue of the stability condition, a family of adaptive laws and a switching law are developed.

- **Robust stability and stabilization of switched linear systems with switching delays**

  We introduce a Lyapunov function to study switched linear systems with switching delays: this Lyapunov function is continuous at switching instants and discontinuous at the instant when the controller and the system mode are matched. Using this Lyapunov function, a novel robust controller is designed that can guarantee a non-weighted $\mathcal{L}_2$ gain for switched linear systems with switching delays.

1.3 Thesis outline

This thesis consists of two parts. Part I focuses on adaptive tracking control of uncertain switched linear systems: here, adaptive control mechanisms to address the tracking problem based on different switching laws are derived. In Part II, uncertain switched linear systems with different kinds of delays are considered, and adaptive and robust stabilization methods are developed. The organization and the relationship between different chapters of this thesis are shown in Fig. 1.2. The content of each chapter is briefly presented as follows.

Chapter 2: The background on control of uncertain switched linear systems is introduced. Specifically, some basic definitions and stability results about switched linear systems that are exploited to develop the control mechanisms are recalled. Moreover, representative stability results for switched linear systems subject to switching delays and time delays are presented. We also revisit the basic ideas of adaptive control of uncertain non-switched linear systems and the state of the art on adaptive control of uncertain switched linear systems.

Chapter 3: A family of new adaptive control schemes for uncertain switched linear systems is developed based on different switching laws that exploit the information of every subsystem, namely mode-dependent dwell time, mode-mode dependent dwell time, and mode-dependent average dwell time switching. These switching laws allow even shorter

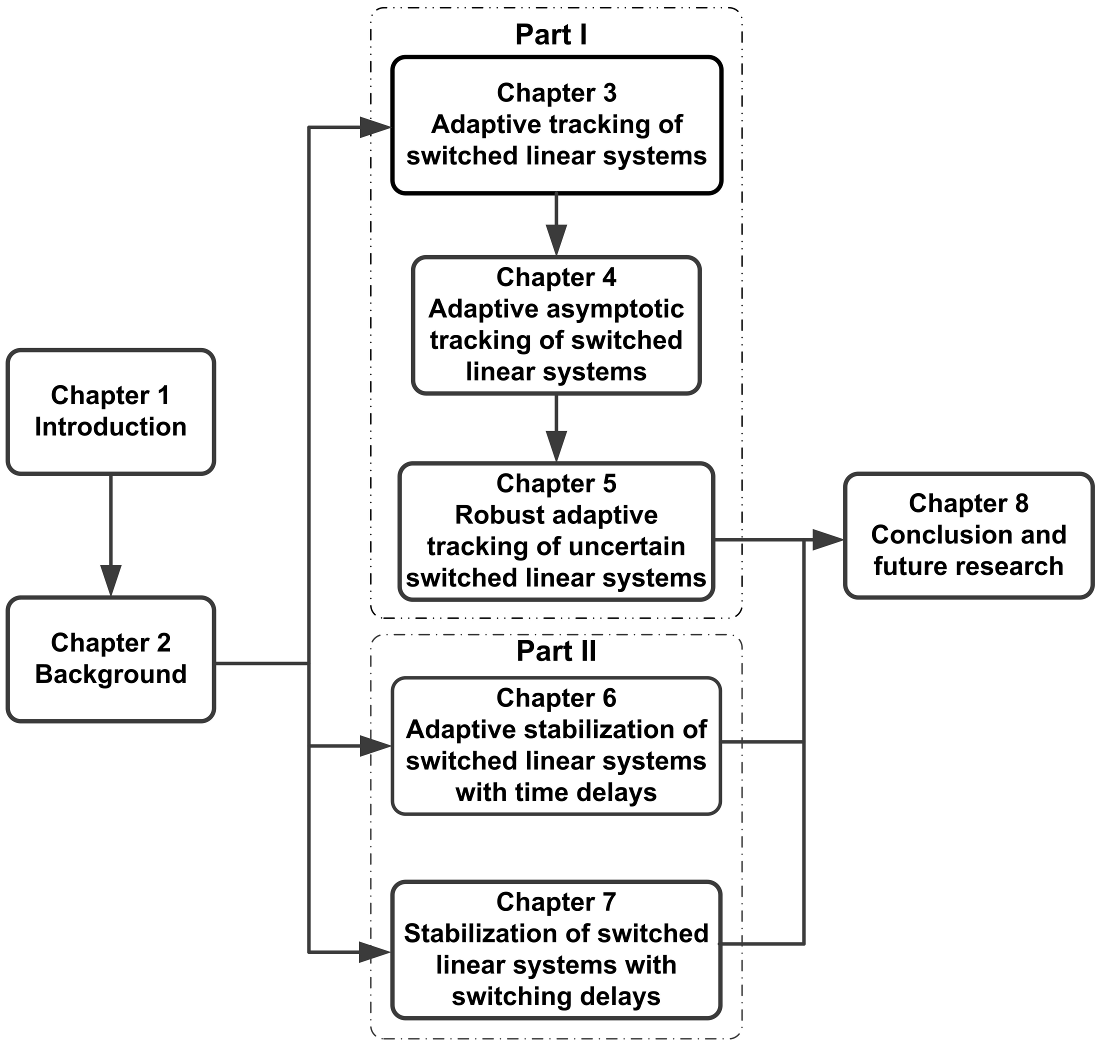

Figure 1.2: Structure of this thesis.

switching intervals than dwell time and average dwell time switching, respectively. Global uniform ultimate boundedness of the switched adaptive closed-loop system is shown, and the bounds on transient and steady-state performance are also presented.

Chapter 4: A Lyapunov function that is decreasing during the intervals between two consecutive switching instants and non-increasing at the switching instants, is exploited to develop a novel model reference adaptive law for uncertain switched linear systems. With this new Lyapunov function, asymptotic stability of the switched adaptive closed-loop system is established for the first time, i.e., the tracking error converges to zero asymptotically, even when no common Lyapunov function for the reference models exists. Furthermore, if the reference input is persistently exciting, we can also guarantee that the parameter estimates of the state-feedback controller converge to the nominal parameters asymptotically. The results in this chapter close an important theoretical gap between adaptive control of non-switched and of switched linear systems.

Chapter 5: In this chapter, the results about adaptive control of switched linear systems in Chapter 3 are extended in the presence of disturbances by developing two robust adaptive control schemes for switched linear systems, namely, the parameter projection method and the leakage method. The switched adaptive closed-loop system is shown to be globally uniformly ultimately bounded, and ultimate bounds for both cases are also given.

Chapter 6: This chapter introduces a new adaptive design for uncertain switched linear systems that can handle impulses of states and discontinuities of time-varying delays. A stability condition is developed with which a new switched adaptive controller is proposed. With the designed adaptive law and a switching law based on mode-dependent dwell time switching, global uniform ultimate boundedness of the closed-loop switched linear system can be guaranteed.

Chapter 7: A Lyapunov function is proposed to study switched linear systems with switching delays. The major idea behind the novel Lyapunov function is the consistency with the fundamental property of switched systems with switching delays. This Lyapunov function is continuous at switching instants and discontinuous at the instant when the controller and the system mode are matched. The structure of the Lyapunov function is exploited to develop novel stability criteria for global asymptotic stability of switched linear systems with switching delays. Most importantly, a non-weighted $\mathcal{L}_2$ gain result is established in presence of external disturbances, whereas the state-of-the-art Lyapunov functions are not shown to be capable of guaranteeing a non-weighted $\mathcal{L}_2$ gain.

Finally, conclusions and recommendations for future work are discussed in Chapter 8.

Chapter 2

Background on Stability and Adaptive Control of Switched Linear Systems

In this chapter, a brief introduction on stability results about switched linear systems involved in the subsequent chapters of this thesis under slow switching is presented. In addition, the general mathematical models of and stability analysis of switched linear systems with switching delays and time-varying delays are introduced. The main results about adaptive tracking control of classic non-switched linear systems are revisited and compared with state-of-the-art results about adaptive control of switched linear systems.

2.1 Stability of switched linear systems under slow switching

In this section, the existing stability conditions of switched linear systems exploited in this thesis without and with considering time delays are presented, respectively.

2.1.1 Switched linear systems without time delays

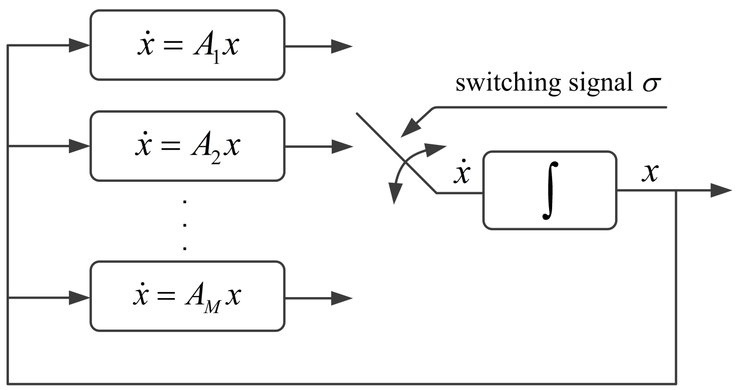

Figure 2.1: The framework of autonomous switched linear systems.

A time-driven switched linear system, as shown in Fig. 2.1, can be mathematically described as

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad t \geq 0 \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the vector of state variables at time t, the switching signal $\sigma(t) : [0,\infty) \to \mathcal{M} := \{1,2\ldots,M\}$ is a piecewise function with M denoting the number of subsystems, and $A_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{M}$, is the dynamics matrix.

First, the following definition of stability is given.

Definition 2.1 *[60] [**Global asymptotic stability**] The switched linear system* (2.1) *is globally asymptotically stable if there exists a class $\mathcal{KL}$ function β such that for all initial conditions and for a given switching signal σ the solution of* (2.1) *satisfies the inequality*

$$|x(t)| \le \beta(|x(0)|, t), \quad t \ge 0. \tag{2.2}$$

In addition, we use the notation $\{(\sigma(t_i),\ t_i)|\ i \in \mathbb{N}\}$ to represent the set of mode-switching instant pairs. The sequence of switch-in instants (entering instants) of subsystem $p,\ \ p \in \mathcal{M}$, is defined by $\{t_l^{p_{\text{in}}}|\ l \in \mathbb{N}^+\}$ and the sequence of switch-out instants (exiting instants) of subsystem $p,\ \ p \in \mathcal{M}$, is defined by $\{t_{l+1}^{p_{\text{out}}}|\ l \in \mathbb{N}^+\}$. Then, the length of the l^{th} active interval of subsystem p is $t_{l+1}^{p_{\text{out}}} - t_l^{p_{\text{in}}}$ for all $l \in \mathbb{N}^+$.

First, the well-known result about asymptotic stability of switched systems based on multiple Lyapunov function is introduced, which will be used to develop stability results using slow switching, i.e., dwell time and average dwell time switching.

Theorem 2.1 *[60] Let* (2.1) *be a finite family of globally asymptotically stable systems, and let V_p, $p \in \mathcal{M}$, be a family of corresponding radially unbounded Lyapunov functions. If there exists a family of positive definite continuous functions W_p, $p \in \mathcal{M}$, with the property that for every pair of switching instants (t_i, t_j), $i < j$, such that $\sigma(t_i) = \sigma(t_j) = p \in \mathcal{M}$ and $\sigma(t_k) \ne p$ for $t_i < t_k < t_j$, we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \le -W_p(x(t_i)).$$

Then, the switched linear system (2.1) *is globally asymptotically stable.*

Theorem 2.1 basically shows that when switching to subsystem $p \in \mathcal{M}$, the value of its corresponding Lyapunov function V_p at one switch-in instant of subsystem p is smaller than that at the previous switch-in instant of subsystem p, as shown in Fig. 2.2. However, this stability result is not easy to be used to develop the slow switching laws because it is difficult to impose the decreasing property for each Lyapunov function V_p at nonsuccessive switching instants [33]. In this regard, the stability condition in Theorem 2.1 is degenerated to the following stability condition.

Theorem 2.2 *[101] Let* (2.1) *be a finite family of globally asymptotically stable systems, and let V_p, $p \in \mathcal{M}$, be a family of corresponding radially unbounded Lyapunov functions. Suppose that for every pair of switching instants (t_{i-1}, t_i), such that $\sigma(t_{i-1}) = q$ and $\sigma(t_i) = p$ with $p, q \in \mathcal{M}$ and $p \ne q$, we have*

$$V_q(x(t_{i-1})) - V_p(x(t_i)) < 0.$$

Then, the switched linear system (2.1) *is globally asymptotically stable.*

Theorem 2.2 essentially claims that when switching from subsystem q to subsystem p at the switching instant t_i, the value of its corresponding Lyapunov function $V_p(t_i)$ is smaller than the value of the Lyapunov function $V_q(t_{i-1})$, as shown in Fig. 2.3. This stability condition will be formulated in Lemma 2.1–Lemma 2.2 to design the switching laws based on dwell time and average dwell time. Let us first introduce the notion of dwell time switching.

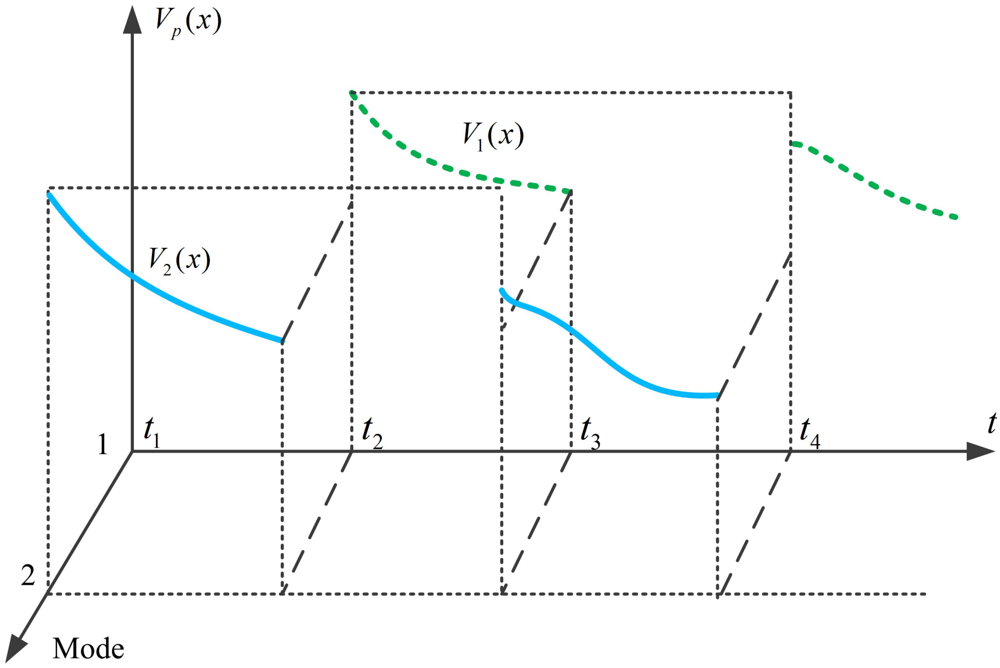

Figure 2.2: Stability condition I: when switching to subsystem 1 at t_4, the value of its corresponding Lyapunov function V_1 at the switch-in instant t_4 of subsystem 1 is smaller than that at the previous switch-in instant t_2 of subsystem 1. This stability condition can be applied to subsystem 2 as well.

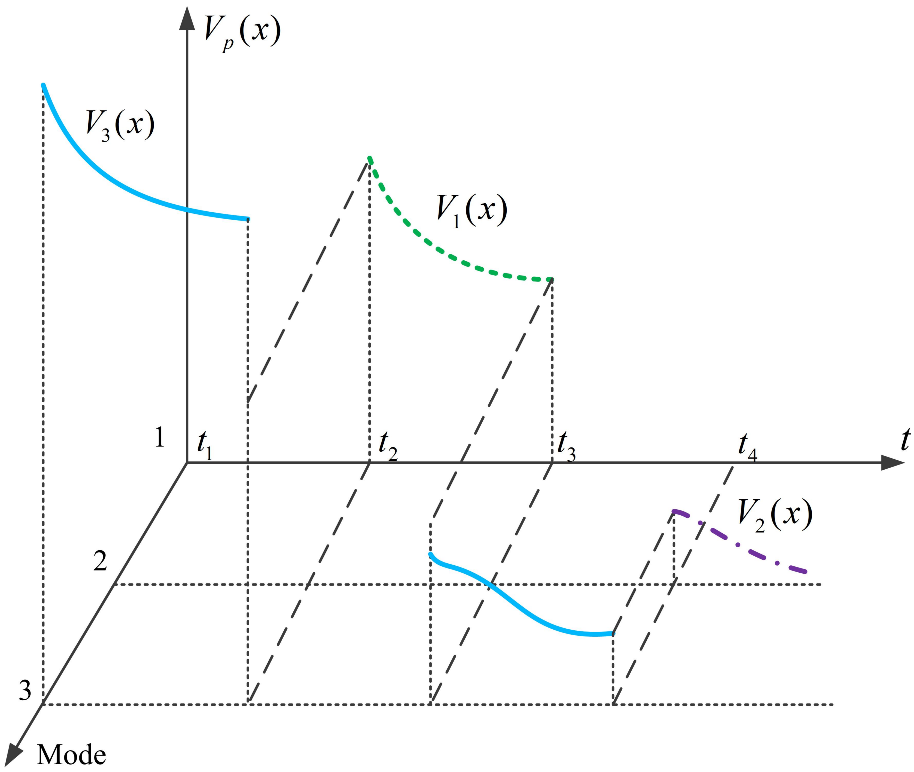

Figure 2.3: Stability condition II: when switching subsystem 3 to subsystem 1 at the switching instant t_2, the value of its corresponding Lyapunov function $V_1(t_2)$ is smaller than the value of the Lyapunov function $V_3(t_1)$ of subsystem 3.

Definition 2.2 *[60] [**Dwell time**] Switching signals are said to belong to the dwell-time admissible set $\mathcal{D}(\tau_{\mathrm{d}})$ if there exists a number $\tau_{\mathrm{d}} > 0$ such that $t_{i+1} - t_i \geq \tau_{\mathrm{d}}$ holds for all $i \in \mathbb{N}^+$. Any positive number τ_{d}, for which these constraints hold for all $i \in \mathbb{N}^+$, is called dwell time.*

An example of a switching signal based on the dwell time switching is presented in Fig. 2.4 where the switching signal admits a dwell time t_{d}.

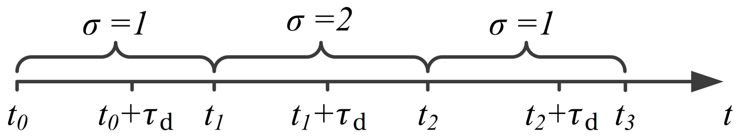

Figure 2.4: A dwell time switching signal.

In what follows, based on the dwell time switching, a well-known result about asymptotic stability of switched linear system (2.1) is introduced using multiple Lyapunov functions [9, 60]. Note that for switched linear systems, quadratic multiple Lyapunov functions $V_p(x) = x^T P_p x$, $p \in \mathcal{M}$ are adopted typically where P_p is a symmetric positive definite matrix.

Lemma 2.1 *[60] Suppose that there exist C^1 functions $V_p : \mathbb{R}^n \to \mathbb{R}$, $p \in \mathcal{M}$, two class $\mathcal{K}_\infty$ functions α_1 and α_2 and two positive numbers $\lambda > 0$ and $\mu \geq 1$ such that we have*

$$\alpha_1(|x|) \leq V_p \leq \alpha_2(|x|) \tag{2.3}$$

and

$$\begin{aligned} \frac{\partial V_p}{\partial x} A_p x &\leq -2\lambda V_p(x) \\ V_p(x) &\leq \mu V_q(x) \end{aligned} \tag{2.4}$$

for all $p, q \in \mathcal{M}$ with $p \neq q$. Then, the switched linear system (2.1) *is globally asymptotically stable for any switching signal $\sigma(\cdot)$ with dwell time*

$$\tau_{\mathrm{d}} > \frac{\ln \mu}{2\lambda}. \tag{2.5}$$

Note that the dwell time (2.5) depends on two key elements: the rate of exponential decrease of the Lyapunov function in between two consecutive switching instants and the finite positive increment of the Lyapunov functions at the switching instants. As one may notice, the rate λ_0 and the increment μ are common to all subsystems p without considering possibly different dynamics of different subsystems. This may give rise to conservative results: a less conservative result than dwell time is proposed in [11, 14, 22], which incorporates the dynamics information of each subsystem, for example the rate of the exponential decrease of the Lyapunov function associated to each subsystem (mode). This extended dwell time is called mode-dependent dwell time, and it is defined as follows.

Definition 2.3 *[22] [**Mode-dependent dwell time**] Switching signals are said to belong to the mode-dependent dwell-time admissible set $\mathcal{D}(\tau_{\mathrm{d}p})$ if for any $p \in \mathcal{M}$ there exists a number $\tau_{\mathrm{d}p} > 0$ such that $t_{l+1}^{p_{\mathrm{out}}} - t_l^{p_{\mathrm{in}}} \geq \tau_{\mathrm{d}p}$ holds for all $l \in \mathbb{N}^+$. Any positive number $\tau_{\mathrm{d}p}$, for which these constraints hold for all $l \in \mathbb{N}^+$, is called mode-dependent dwell time.*

Using the concept of mode-dependent dwell time, the following stability result is given.

Lemma 2.2 *[11, 22] Suppose that there exist C^1 functions $V_p : \mathbb{R}^n \to \mathbb{R}$, two class $\mathcal{K}_\infty$ functions α_1 and α_2 and a family of positive numbers $\lambda_p > 0$ and $\mu_p \geq 1$ such that we have*

$$\alpha_1(|x|) \leq V_p \leq \alpha_2(|x|) \tag{2.6}$$

and

$$\begin{aligned} \frac{\partial V_p}{\partial x} A_p x &\leq -2\lambda_p V_p(x) \\ V_p(x) &\leq \mu_p V_q(x) \end{aligned} \tag{2.7}$$

for all $p, q \in \mathcal{M}$ with $p \neq q$. Then, the switched linear system (2.1) *is globally asymptotically stable for any switching signal $\sigma(\cdot)$ with mode-dependent dwell time*

$$\tau_{\mathrm{d}p} > \frac{\ln \mu_p}{2\lambda_p}. \tag{2.8}$$

Since pairs of positive numbers λ_p and μ_p representing information of every subsystem are used, the mode-dependent dwell time (2.8) admits a larger class of switching signals than dwell time (2.5) does. As an alternative way to reduce the conservativeness of (2.5), a stability condition based on dwell time switching has been proposed without explicitly involving the aforementioned two crucial properties of the Lyapunov functions. Notably, to date, Lemma 2.3 gives rise to the least conservative result about asymptotic stability of switched linear system with dwell time switching.

Lemma 2.3 *[33] Assume that for $\tau_{\mathrm{d}} > 0$, there exists a collection of symmetric positive definite matrices $P_1, \ldots, P_M \in \mathbb{R}^{n\times n}$ such that*

$$A_p^T P_p + P_p A_p < 0, \quad \forall p \in \mathcal{M}$$

and

$$e^{A_p^T \tau_{\mathrm{d}}} P_q e^{A_p \tau_{\mathrm{d}}} - P_p < 0, \quad \forall p \neq q \in \mathcal{M}. \tag{2.9}$$

Then, the switched system (2.1) *is globally asymptotically stable for any switching signal $\sigma(\cdot)$ with dwell time τ_{d}.*

Another well-known slow switching law is based on average dwell time that can relax the concept of dwell time by allowing fast switchings provided that they are compensated by sufficiently slow switchings. In other words, the dwell time (2.5) is realized in an average sense. In what follows, the definition of average dwell time is introduced.

Definition 2.4 *[60] [**Average dwell time**] Let us denote the number of discontinuities of a switching signal $\sigma(\cdot)$ over an interval (t, T) by $N_\sigma(T, t)$. We say that $\sigma(\cdot)$ has average dwell time τ_{a} if there exist two positive numbers N_0 and τ_{a} such that*

$$N_\sigma(T, t) \leq N_0 + \frac{T - t}{\tau_{\mathrm{a}}}. \tag{2.10}$$

The switching signal admitting a dwell time t_{d} in Fig. 2.4 is relaxed into a switching signal admitting an average dwell time t_{a}, as shown in Fig. 2.5, which shows that some of the switching intervals are allowed to be smaller than t_{d} and they are compensated by longer switching intervals.

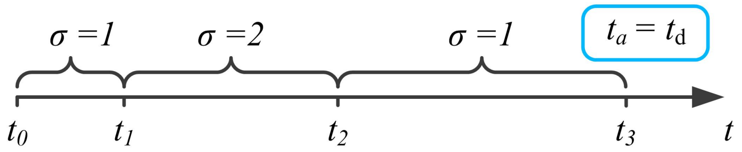

Figure 2.5: The switching signal based on dwell time in Fig. 2.4 is relaxed to the switching signal based on average dwell time.

Similar with dwell time, the concept of average dwell time can be relaxed by incorporating information of each subsystem, which gives rise to the concept of mode-dependent average dwell time as follows.

Definition 2.5 *[31, 154] [**Mode-dependent average dwell time**] For a switching signal $\sigma(\cdot)$, let $N_{\sigma p}(t_1, t_2)$, $t_2 \geq t_1 \geq 0$, $p \in \mathcal{M}$, be the number of times that subsystem p is activated over the interval $[t_1, t_2)$ and let $T_p(t_1, t_2)$ denote the total running time of subsystem p over the interval $[t_1, t_2]$, $p \in \mathcal{M}$. We say that $\sigma(\cdot)$ has a mode-dependent average dwell time (MDADT) $\tau_{\mathrm{a}p}$ if for any $p \in \mathcal{M}$ there exist positive numbers N_{0p} and $\tau_{\mathrm{a}p}$ such that*

$$N_{\sigma p}(t_1, t_2) \leq N_{0p} + \frac{T_p(t_1, t_2)}{\tau_{\mathrm{a}p}}, \quad \forall t_2 \geq t_1 \geq 0 \tag{2.11}$$

where N_{0p} are called mode-dependent chatter bounds.

The global asymptotic stability results about switched linear system (2.1) given by Lemma 2.1 based on dwell time and by Lemma 2.2 based on mode-dependent dwell time can directly be applied to average dwell time and mode-dependent average dwell time [154]. It is important to mention that different selections of N_0 and N_{0p} do not influence the asymptotic stability, but only affect the overshoot of the switched systems: a large N_0 or N_{0p} leads to a large overshoot [60].

Beyond the scope of the slow switching schemes using (mode-dependent) dwell time and (mode-dependent) average dwell time, a less conservative stability analysis for switched systems have been introduced in [50] which does not involve bounds on the number of switches over time intervals. However, the price paid for the improvement over the slow switching schemes is the significant increase of computational complexity.

As a final remark, since the aforementioned stability results, without exception, stem from quadratic Lyapunov functions using constant symmetric positive definite matrices, a positive number $\mu > 1$ is bound to be involved explicitly (cf. Lemma 2.1 and Lemma 2.2) or implicitly (cf. Lemma 2.3) given that no common Lyapunov function exists. In other words, the Lyapunov functions increase at some switching instants, which will be shown to be the very obstacle in Chapter 4 that prevents adaptive control of switched systems from achieving asymptotic stability.

2.1.2 Switched linear systems with time-varying delays

Switched systems with time-delay states are natural generalizations of switched systems, as time-varying delay of the system state is a common problem in hybrid systems [29, 55, 65, 98, 105, 113, 146, 149, 153]. Switched linear system with time-varying delays can be described as follows

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t) + E_{\sigma(t)}x(t-d(t)),\\ x(\theta) &= \phi(\theta), \quad \theta \in [-\tau, 0]\end{aligned} \tag{2.12}$$

where $A_p \in \mathbb{R}^{n\times n}$, $E_p \in \mathbb{R}^{n\times n}$, $\phi \in \mathbb{R}^n$ is the initial function, and $\tau := \sup_{t>t_0} d(t)$.

Stability analysis of switched linear systems (2.12) with time-varying delays has been intensively investigated [17, 39, 52, 53, 66, 98, 99, 105]. Two families of techniques for non-switched time-delay systems, which are Krasovskii-based techniques and Razumikhin-based techniques, have been extended by incorporating multiple Lyapunov functions. The following widely-used Lyapunov-Krasovskii function was proposed in [98]

$$V(t) = x^T(t)P_{\sigma(t)}x(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)e^{\alpha(s-t)}Z_{\sigma(t)}\dot{x}(s)dsd\theta + \int_{t-d(t)}^{t} x^T(s)e^{\alpha(s-t)}Q_{\sigma(t)}x(s)ds \tag{2.13}$$

where P_p, Z_p, and $Q_p \in \mathbb{R}^{n\times n}$, $\forall p \in \mathcal{M}$ are symmetric positive definite matrices. Making use of the Lyapunov-Krasovskii function (2.13) and of its revised ones, various stability results about switched time-delay linear system have been developed [17, 39, 52, 53, 66, 98, 99, 105]. One of the well-established stability results is introduced as follows.

Theorem 2.3 *[98] For a given positive number $\alpha > 0$, suppose that there exist symmetric positive definite matrices P_p, Q_p, Z_p,*

$$X = \begin{bmatrix} X_{11}^p & X_{12}^p \\ * & X_{22}^p \end{bmatrix} \geq 0 \tag{2.14}$$

and any matrices Y_p, T_p, $p \in \mathcal{M}$ with appropriate dimensions and a positive number $\mu \geq 1$ such that

$$\begin{bmatrix} \varphi_{11} & \varphi_{12} & \tau A^T Z \\ * & \varphi_{22} & \tau E^T Z \\ * & * & -\tau Z \end{bmatrix} < 0, \quad \begin{bmatrix} X_{11}^p & X_{12}^p & Y_p \\ * & X_{22}^p & T_p \\ * & * & e^{-\alpha\tau} Z \end{bmatrix} \geq 0 \tag{2.15}$$

where

$$\begin{aligned} &P_p \leq \mu P_q, \quad Q_p \leq \mu Q_q, \quad Z_p \leq \mu Z_q \\ &\varphi_{11}^p = A_p^T P_p + P_p A_p + Y_p + Y_p^T + Q_p + \tau X_{11}^p + \alpha P_p \\ &\varphi_{22}^p = -T_p - T_p^T - (1-d)e^{-\alpha\tau}Q_p + \tau X_{22}^p. \end{aligned}$$

Then, the switched linear system with time-varying delays (2.12) *is globally asymptotically stable for any switching signal $\sigma(\cdot)$ with average dwell time*

$$T_{\mathrm{a}} > T_{\mathrm{a}}^* = \frac{\ln\mu}{\alpha}. \tag{2.16}$$

In what follows, the Razumikhin-based result about asymptotic stability of system (2.12) is presented. First, an introduction of Lyapunov-Razumikhin function for non-switched time-delay systems is given in the following lemma.

Lemma 2.4 *[131] Suppose $u, v, \omega, \mathscr{P}: \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, non-decreasing functions with $u(0) = v(0) = 0$, $u(s), v(s), \omega(s), \mathscr{P}(s)$ positive for $s > 0$, $\mathscr{P}(s) > s$, and $v(\cdot)$ strictly increasing. If* *there exists a continuous function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$u(\|x(t)\|) \le V(t,x) \le v(\|x(t)\|), \quad t \ge 0, \quad x \in \mathbb{R}^n \tag{2.17}$$

and

$$\dot{V}(t,x(t)) \le -\omega(\|x(t)\|) \tag{2.18}$$

if

$$V(t+\theta, x(t+\theta)) < \mathscr{P}(V(t,x(t))) \tag{2.19}$$

then, the general retarded functional differential equation $\dot{x}(t) = f(t,x(t),x(t+\theta))$ *is uniformly asymptotically stable.*

By applying Lemma 2.4 to switched time-delay linear systems, the multiple Lyapunov functions $V_p = x^T P_p x$, $p \in \mathcal{M}$ are used, where the following conditions hold

$$\kappa_p \|x\|^2 \le V_p \le \bar{\kappa}_p \|x\|^2 V_p(t+\theta, x(t+\theta)) < \mathscr{P}_p(V_p(t,x(t)))$$

and

$$\dot{V}_p \le -x^T S_p x$$

with a family of positive numbers κ_p, $\bar{\kappa}_p$, $\mathscr{P}_p$, and symmetric positive definite matrices S_p and P_p. Define $\lambda := \max_{p\in\mathcal{M}} \bar{\kappa}_p/\kappa_p$, and $\mu := \max_{p\in\mathcal{M}} \bar{\kappa}_p/\omega_p$, where $\omega_p > 0$ is the smallest singular value of S_p for all $p \in \mathcal{M}$. Then, the asymptotic stability result about switched linear systems with time delays is derived in [131] as follows.

Theorem 2.4 *[131] Let the dwell time be defined by* $\tau_\mathrm{d} := T^* + \tau$, *where*

$$T^* := \lambda\mu \left\lfloor \frac{\lambda - 1}{\bar{\mathscr{P}} - 1} + 1 \right\rfloor$$

with $\bar{\mathscr{P}} := \min_{p\in\mathcal{M}} \mathscr{P}_p > 1$, $\lfloor\cdot\rfloor$ *being the floor integer function. Then, the switched time-delay system* (2.12) *is globally asymptotically stable for any switching signal* $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$.

It is apparent that the stability results based on the Lyapunov-Krasovskii function need continuity of the time-varying delays, since the derivative of the time-varying delays is involved. On the other hand, the Lyapunov-Razumikhin function may handle discontinuous delays, but it needs the existence of the constant (function) $\mathscr{P}_p$ for linear systems (nonlinear systems).

2.1.3 Switched linear systems with switching delays

The controller design based on the stability conditions has been investigated intensively [10, 19, 21, 33, 61, 94, 100, 123, 140, 148], where, typically, the focus is on synchronously switched linear systems, an ideal case in which the controller is assumed to switch synchronously with the system mode. However, due to the delay between a mode change and the activation of its corresponding controller, or due to the time needed to detect switching of system mode, nonzero time intervals are present during which system modes and controller modes are mismatched [30, 69, 111, 112, 119, 132, 143, 144]. These time intervals all called *unmatched intervals*, and the counterparts when the subsystem and its corresponding controller are matched are called *matched intervals*. A switched system with switching delays is shown in Fig. 2.6, where there exists a switching delay $\Delta\tau(t)$, $t \in \mathbb{R}_{\geq 0}$, between the activation of the subsystem and the activation of the controller. A typical example in engineering practice can be seen in teleoperation, e.g. [73, 74, 116].

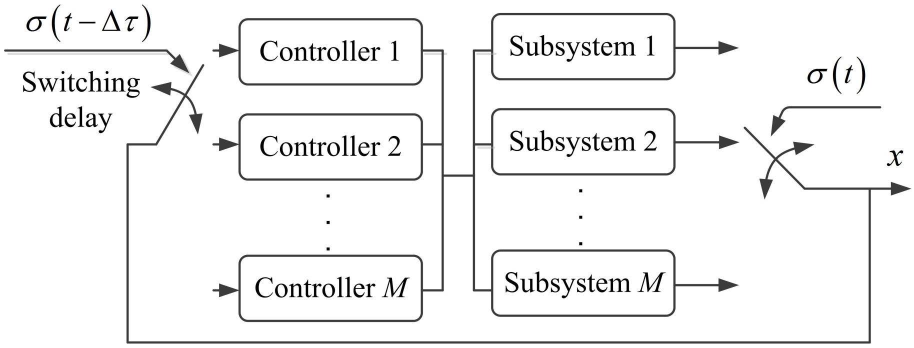

Figure 2.6: The framework of switched system with switching delays.

The switched linear system with switching delays can be formulated mathematically as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \tag{2.20}$$

with the input $u(t) = K_{\sigma(t-\Delta\tau(t))}x(t)$, which gives rise to

$$\dot{x}(t) = \left(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t-\Delta\tau(t))}\right)x(t) \tag{2.21}$$

with $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{m\times n}$. In what follows, a lemma is used widely to study asymptotic stability of switched linear systems with switching delays.

Lemma 2.5 *[143] Let $\alpha > 0$, $\beta > 0$ and $\mu > 1$ be given constants. Suppose that there exist C^1 functions $V_{\sigma(t)}: \mathbb{R}^n \to \mathbb{R}$, and two class K_∞ functions κ_1 and κ_2 such that*

$$\kappa_1(|x|) \leq V_p(x) \leq \kappa_2(|x|), \quad p \in \mathcal{M}$$

$$\dot{V}_p(x) \leq \begin{cases} -\alpha V_p(x), & \textit{for matched intervals} \\ \beta V_p(x), & \textit{for unmatched intervals} \end{cases}$$

and

$$V_p(x) \leq \mu V_q(x)$$

for any $p \neq q \in \mathcal{M}$. Then, the switched linear system with switching delays (2.21) *is globally asymptotically stable for any switching signal with average dwell time*

$$\tau_{\mathrm{a}} \geq \frac{\Delta\tau_{\sup}(\alpha+\beta) + \ln\mu}{\alpha}$$

with $\Delta\tau_{\sup} = \sup\{\Delta\tau(t) | t \geq 0\}$.

Note that the Lyapunov functions proposed in Lemma 2.5 extend the multiple Lyapunov functions in Lemma 2.1–2.3: the Lyapunov functions in Lemma 2.5 are allowed to increase during the unmatched intervals which are compensated by the decrease during the matched intervals. In view of this, the stability condition in Lemma 2.5 exploits explicitly the information of the exponential decrease and increase in between two consecutive switching instants and the possible increment of the Lyapunov functions.

2.2 Adaptive control of switched linear systems

Before introducing the state of the art on adaptive control of switched linear systems, classical adaptive control of uncertain non-switched linear systems is revisited and the control scheme is shown in Fig. 2.7. The desired system dynamics are described by a reference model which is a linear time-invariant system driven by a reference signal. The control law is then designed to guarantee that the uncertain closed-loop system can track the states of the reference model [103].

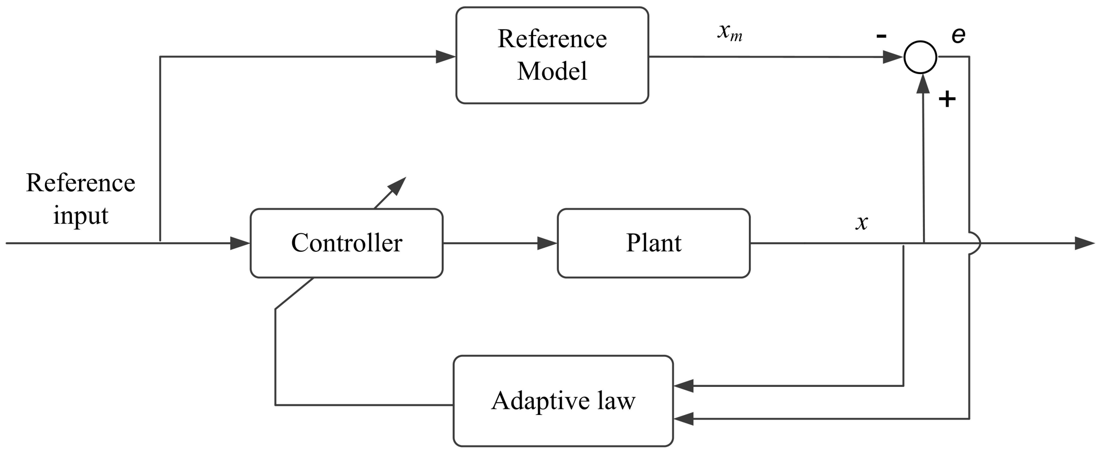

Figure 2.7: The framework of adaptive control of non-switched systems.

Before presenting the result of adaptive control of non-switched linear systems, the following important definition is given for the reference signal.

Definition 2.6 *[43] [**Persistently exciting condition**] Consider a signal vector v generated as $v(t) = H(s)\xi(t)$[1] where $\xi \in \mathbb{R}$, and $H(s)$ is a vector whose elements are transfer functions that are strictly proper with stable poles. If the complex vectors $H(j\omega_1),\ldots,H(j\omega_n)$ are linearly independent on the complex space $\forall\omega_1,\ldots,\omega_n$, where $\omega_i \neq \omega_j$ for $i \neq j$, then we say v is persistently exciting if and only if the spectrum of ξ contains at least $n/2$ different nonzero frequencies.*

In what follows, important results about adaptive tracking control of non-switched system with state feedback are presented [103, 104]. Consider a general linear time-invariant plant in the state-space form

$$\dot{x}(t) = Ax(t) + bu(t) \tag{2.22}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are an unknown constant matrix and an unknown constant vector. The reference signal is generated by

$$\dot{x}_\mathrm{m}(t) = A_\mathrm{m}x_\mathrm{m}(t) + b_\mathrm{m}r(t) \tag{2.23}$$

where $A_\mathrm{m} \in \mathbb{R}^{n\times n}$ and $b_\mathrm{m} \in \mathbb{R}^n$ are a known constant matrix and a known constant vector with A_m Hurwitz, and the reference input $r(t)$ is bounded. Suppose that the following matching

[1]A streamlined notation is used to denote the filtering action of a linear system.

conditions hold

$$A + bk^{*T} = A_{\mathrm{m}}, \quad bl^* = b_{\mathrm{m}}$$

where $k^* \in \mathbb{R}^n$ and $l^* \in \mathbb{R}$ are the nominal parameters such that the nominal controller $u(t) = k^{*T}x(t) + l^* r(t)$ can make the unknown closed-loop system track the reference model asymptotically. Then, with the assumption about the knowledge of the sign of l^*, the following adaptive laws for the unknown parameters k^* and l^* are used

$$\dot{k}(t) = -\operatorname{sgn}[l^*]\Gamma x(t)e^T(t)Pb_{\mathrm{m}}$$
$$\dot{l}(t) = -\operatorname{sgn}[l^*]\gamma r(t)e^T(t)Pb_{\mathrm{m}}$$

via the Lyapunov function

$$V(t) = e^T(t)Pe(t) + \frac{1}{|l^*|}\tilde{k}^T(t)\Gamma^{-1}\tilde{k}(t) + \frac{1}{|l^*|}\gamma^{-1}\tilde{l}^2(t)$$

where $\tilde{k} := k - k^*$, $\tilde{l} := l - l^*$ are parameter estimation errors, $\Gamma \in \mathbb{R}^{n\times n}$ and $\gamma \in \mathbb{R}$ are a positive definite matrix and a positive number, respectively, and $P \in \mathbb{R}^{n\times n}$ is a constant matrix such that

$$A_{\mathrm{m}}^T P + PA_{\mathrm{m}} < 0.$$

The resulting controller $u(t) = k^T(t)x(t) + l(t)r(t)$ can guarantee that the uncertain system (2.22) asymptotically tracks the reference model (2.23), i.e., $e(t) := x(t) - x_{\mathrm{m}}(t) \to 0$ for $t \to \infty$. Furthermore, when the reference input r is of persistent excitation, it follows $k(t) \to k^*$ and $l(t) \to l^*$ as $t \to \infty$.

Adaptive approaches for uncertain non-switched linear systems have been investigated intensively to improve the performance of robust approaches over large non-polytopic uncertainties [4, 51, 75, 103]. On the other hand, adaptive control of uncertain switched linear systems is more challenging. This is because not only an adaptive law should be developed to estimate the unknown parameters, but also a switching law should be carefully designed to guarantee the stability of the closed-loop system. The basic framework of adaptive control of switched systems is given in Fig. 2.8, where adaptive control of non-switched systems in Fig. 2.7 is extended to switched systems case.

For convenience, we introduce the uncertain switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t) \tag{2.24}$$

where $A_p \in \mathbb{R}^{n\times n}$ and $b_p \in \mathbb{R}^n$ are an unknown matrix and an unknown vector for all $p \in \mathcal{M}$, and $\sigma(t) : [0,\infty) \to \mathcal{M} := \{1, 2 \ldots, M\}$ is the switching signal. The reference dynamics are represented by

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}\sigma(t)}x_{\mathrm{m}}(t) + b_{\mathrm{m}\sigma(t)}r(t) \tag{2.25}$$

where $A_{\mathrm{m}p} \in \mathbb{R}^{n\times n}$ and $b_{\mathrm{m}p} \in \mathbb{R}^n$ are a known matrix and a known vector with $A_{\mathrm{m}p}$ Hurwitz for all $p \in \mathcal{M}$, and the reference input $r(t)$ is bounded. Suppose that the following matching conditions hold

$$A_p + b_p k_p^{*T} = A_{\mathrm{m}p}, \quad b_p l_p^* = b_{\mathrm{m}p}$$

where $k_p^* \in \mathbb{R}^n$ and $l_p^* \in \mathbb{R}$, $p \in \mathcal{M}$, are the nominal parameters.

In what follows, we introduce two representative results using dwell time and average dwell time switching, respectively.

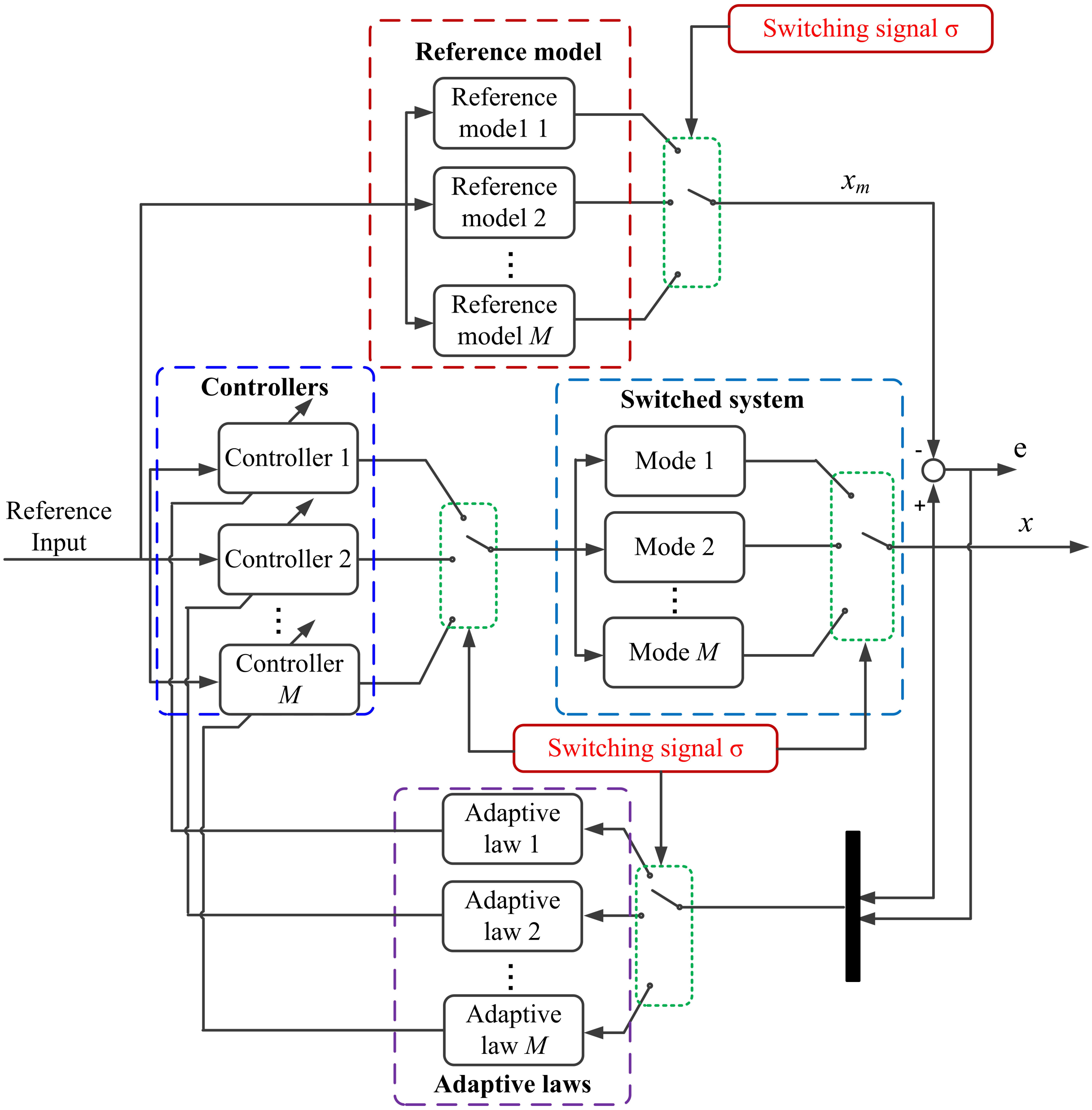

Figure 2.8: The framework of adaptive control of switched systems.

Assumption 2.1 *The sign of l_p^* for any $p \in \mathcal{M}$ should be known.*

Assumption 2.2 *The possible upper and lower bounds of the unknown parameters k_p and l_p for any $p \in \mathcal{M}$ should be known.*

<u>*Result 1*</u>: With the assumptions 2.1 and 2.2, Sang and Tao [91] proposed a switching law based on the dwell time and adaptive laws with parameter projection for subsystem p, $p \in \mathcal{M}$

$$\begin{aligned} \dot{k}_p(t) &= -\mathrm{sgn}[l_p^*]\Gamma_p x(t) e^T(t) P_p b_{\mathrm{mp}} + f_{xp}(t) \\ \dot{l}_p(t) &= -\mathrm{sgn}[l_p^*]\gamma_p r(t) e^T(t) P_p b_{\mathrm{mp}} + f_{rp}(t) \end{aligned} \tag{2.26}$$

where the symmetric positive definite matrix P_p is the solution to

$$A_{\mathrm{mp}}^T P_p + P_p A_{\mathrm{mp}} = -Q_{\mathrm{mp}}, \quad p \in \mathcal{M}$$

for a given symmetric positive definite matrix Q_p, f_{xp} and f_{rp}, $p \in \mathcal{M}$ are the parameter projection laws that keep the parameter estimates k_p and l_p bounded [103]. Note that the adaptive laws (2.26) are developed using the Lyapunov function

$$V_p(t) = e^T(t)P_p e(t) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\gamma_p^{-1}\tilde{l}_p^2(t)$$

where $\tilde{k}_p := k_p - k_p^*$ and $\tilde{l}_p := l_p - l_p^*$, $p \in \mathcal{M}$, are parameter estimation errors.

Find positive numbers $a_{\mathrm{m}p}$ and $\lambda_{\mathrm{m}p}$ such that $\|e^{A_{\mathrm{p}}t}\| \le a_{\mathrm{m}p}e^{-\lambda_{\mathrm{m}p}t}$ for all $p \in \mathcal{M}$. Define $a_{\mathrm{m}} = \max_{p\in\mathcal{M}} a_{\mathrm{m}p}$, $\lambda_{\mathrm{m}} = \min_{p\in\mathcal{M}} \lambda_{\mathrm{m}p}$, $\alpha = \max_{p\in\mathcal{M}} \lambda_{\max}(P_p)$, and $\beta = \min_{p\in\mathcal{M}} \lambda_{\min}(P_p)$. Then, the stability result in [91] is given explicitly.

Lemma 2.6 *[91] With the adaptive laws* (2.26) *and the switching signal with dwell time*

$$\tau_{\mathrm{d}} = \alpha(1+\kappa)\ln(1+\mu\Delta A_{\mathrm{m}}), \quad \kappa > 0 \tag{2.27}$$

where $\Delta A_{\mathrm{m}} = \max_{p,q\in\mathcal{M}} \|A_{mp} - A_{mq}\|$, *and* $\mu = a_{\mathrm{m}}^2 \max_{p\in\mathcal{M}} \|P_p\|/(\lambda_{\mathrm{m}}\beta)$, *all signals in the closed-loop system are bounded, and the tracking error is bounded in sense that*

$$\int_t^{t+T} e^T(\tau)e(\tau)d\tau \le \mu\Delta_{A_{\mathrm{m}}} c_0 \frac{T}{T_0} + c_1, \quad t \ge 0, \quad T > 0 \tag{2.28}$$

where $c_1 = (1+\mu\Delta_{A_{\mathrm{m}}})c_0$ *for some* $c_0 > 0$.

Two crucial properties of the Lyapunov function are exploited in [91]: an exponential rate of decrease during the active intervals between two consecutive switching instants and a bounded increment at switching instants. Because of this, asymptotic stability can be guaranteed only in the presence of a common Lyapunov function for the reference models. For general settings when no common Lyapunov function exists, the control method proposed in [91] can only guarantee (non-asymptotic) stability of the closed-loop switched system and that the tracking error is bounded in a mean square sense (cf. (2.28)).

Furthermore, parameter projection is a necessary tool to keep the estimates bounded, even in the absence of any disturbance. These results are not consistent with the well-known results about adaptive tracking control for classical non-switched systems, where parameter projection is not needed in the noiseless case, and asymptotic tracking can be guaranteed [44, 103].

Result 2: With the assumptions 2.1 and 2.2, the adaptive laws similar with (2.26) are used in [117] to address parametric uncertainties of multiple-input switched linear systems with average dwell time switching

$$\begin{aligned} \dot{K}_p^T(t) &= -S_p^T B_{\mathrm{m}p}^T P_p e(t)x^T(t) + F_{xp}^T(t) \\ \dot{L}_p(t) &= -S_p^T B_{\mathrm{m}p}^T P_p e(t)r^T(t) + F_{rp}(t) \end{aligned} \tag{2.29}$$

where K_p and L_p are the estimates of controller parameters for multiple-input switched linear systems, S_p are square matrices of compatible dimensions depending on the nominal parameters L_p^*, F_{xp} and F_{rp} are the projection laws [103] that keep the parameter estimates stay in the known bounds.

Definition 2.7 (Global uniform ultimate boundedness) *[44] The uncertain switched linear* *system* (3.1) *under switching signal* $\sigma(\cdot)$ *is globally uniformly ultimately bounded (GUUB) if there exists a finite positive number* b_{T} *such that there exists a finite* $\mathscr{T}$ *such that* $\|x(t)\| \leq b_{\mathrm{T}}$ *for all* $t \geq t_0 + \mathscr{T}$. *Any positive number* b_{T} *for which this condition holds is called ultimate bound.*

The stability result proposed in [117] is generalized in the following lemma, where only global uniform ultimate boundedness of the closed-loop adaptive system is shown and the transient and steady-state performance of the tracking error are not investigated. This result apparently leads to a deviation from the result about adaptive control of non-switched linear systems.

Lemma 2.7 *[117] With the adaptive laws* (2.29) *and the switching signal with a proper design of average dwell time, all signals in the closed-loop systems are bounded, and the tracking error is globally uniformly ultimately bounded.*

2.3 Concluding remarks

In this chapter, some stability conditions for switched linear systems using multiple Lyapunov function have been presented which will be involved in the next chapters. In particular, asymptotic stability results based on dwell time and average dwell time switching have been given, together with their extended notions of mode-dependent dwell time and mode-dependent average dwell time switching. In addition, representative stability results about switched linear systems with switching delays and time-varying delays have been introduced. Finally, the classic adaptive control of non-switched linear systems is revisited, which is followed by an introduction of the framework for adaptive control of switched linear systems and its state-of-the-art stability results.

Part I

Adaptive Tracking Control of Uncertain Switched Linear Systems

Chapter 3

Adaptive Tracking of Switched Linear Systems with Extended Dwell Time and Average Dwell Time

In this chapter, two families of adaptive tracking control schemes for uncertain switched linear systems are developed based on mode-dependent dwell time and mode-dependent average dwell time switching, which exploit the information of every subsystem. Furthermore, to address the cases in which the next subsystem to be switched to is known, we propose a new time-driven switching approach: mode-mode-dependent dwell time switching. The adaptive controller, consisting of a switching law and adaptive laws, guarantees global uniform ultimate boundedness of the switched adaptive closed-loop system.

Parts of the research presented in this chapter have been published in [136, 137].

3.1 Introduction

To date, some research has been conducted on adaptive tracking of uncertain switched linear systems based on dwell time [89–92] and on average dwell time [117, 118]. However, to the best of the author's knowledge, not much attention has been paid on reducing the conservativeness of standard slow switching laws by exploiting the information of each subsystem. Conservativeness is interpreted as the time interval required to switch from one mode to another (which should be as short as possible to approach arbitrarily fast switching). In addition, the transient and steady-state performance of the tracking error have not been studied, which are important aspects in adaptive closed-loop systems. To this end, by extending the results in [91, 117], two families of adaptive tracking control schemes for uncertain switched systems are developed based on mode-dependent dwell time (MDDT) switching laws and mode-dependent average dwell time (MDADT) switching laws, by exploiting the information of every subsystem. Furthermore, to address the cases in which the next subsystem to be switched to is known, such as in automobile power train [130], power converters [67], and other applications, we propose a new time-driven switching approach: mode-mode-dependent dwell time (MMDDT) switching. It not only exploits the information of the current subsystem but also of the next subsystem. This allows even shorter switching intervals than MDDT. Time intervals in switching laws based on MDADT are generally smaller than time intervals based on ADT and DT. In addition, globally uniformly ultimately bounded (GUUB) stability of the switched system via the proposed adaptive tracking control schemes is shown. Notably, computable transient and steady-state performance bounds are derived, where an upper bound and the ultimate bound characterizing the global uniform ultimate boundedness of the tracking error are used as a measure for the transient and steady-state performance, respectively.

The chapter is organized as follows. In Section 3.2, the mathematical model and problem formulation are given. Next, the adaptive control schemes with a family of adaptive laws and three different switching laws are proposed in Section 3.3. The transient and steady-state performance analysis of the tracking error is introduced in Section 3.4. An example is used to illustrate the proposed methods in Section 3.5. The chapter is concluded in Section 3.6.

3.2 Problem statement

Consider the uncertain switched multiple-input linear system given by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad \sigma(t) \in \mathcal{M} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, and the switching signal $\sigma : [0,\infty) \to \mathcal{M} := \{1, 2, \ldots, M\}$ is a piecewise function with M denoting the number of subsystems. We say a subsystem $p \in \mathcal{M}$ is uncertain when the matrices $A_p \in \mathbb{R}^{n\times n}$ and $B_p \in \mathbb{R}^{n\times m}$ are unknown.

A family of switched reference models representing the desired behavior of each subsystem is given as follows:

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}\sigma(t)}x_{\mathrm{m}}(t) + B_{\mathrm{m}\sigma(t)}r(t), \quad \sigma(t) \in \mathcal{M} \tag{3.2}$$

where $x_{\mathrm{m}}(t) \in \mathbb{R}^n$ is the desired state vector, and $r(t) \in \mathbb{R}^m$ is a bounded reference input. The matrices $A_{\mathrm{m}p} \in \mathbb{R}^{n\times n}$ and $B_{\mathrm{m}p} \in \mathbb{R}^{n\times m}$ are known and $A_{\mathrm{m}p}$, $p \in \mathcal{M}$, are Hurwitz matrices. The nominal state feedback controllers that make the switched system behave like the reference model are given as $u(t) = K^{*T}_{\sigma(t)}(t)x(t) + L^*_{\sigma(t)}(t)r(t)$, where $K^*_p \in \mathbb{R}^{n\times m}$ and $L^*_p \in \mathbb{R}^{m\times m}$, $p \in \mathcal{M}$, are nominal parameters, which can be calculated by

$$A_p + B_p K_p^{*T} = A_{\mathrm{m}p}, \quad B_p L_p^* = B_{\mathrm{m}p}, \quad p \in \mathcal{M}. \tag{3.3}$$

Since A_p and B_p are unknown, we cannot obtain K^*_p and L^*_p from (3.3). Hence, the state feedback controllers are designed as

$$u(t) = K^T_{\sigma(t)}(t)x(t) + L_{\sigma(t)}(t)r(t) \tag{3.4}$$

where K_p and L_p are the estimates of K^*_p and L^*_p, $p \in \mathcal{M}$, respectively, which are updated by some adaptive laws to be explained in the next section.

In addition, the tracking error is defined as $e(t) = x(t) - x_{\mathrm{m}}(t)$. Substituting (3.4) into (3.1), and subtracting (3.2), we have the following dynamics of the tracking error:

$$\dot{e}(t) = A_{\mathrm{m}\sigma(t)}e(t) + B_{\sigma(t)}(\tilde{K}^T_{\sigma(t)}(t)x(t) + \tilde{L}_{\sigma(t)}(t)r(t)) \tag{3.5}$$

where $\tilde{K}_p = K_p - K^*_p$ and $\tilde{L}_p = L_p - L^*_p$, $p \in \mathcal{M}$ are the parameter estimation errors.

Definition 3.1 *(**Mode-mode-dependent dwell time**) The switching signal $\sigma(\cdot)$ is said to have mode-mode-dependent dwell time (MMDDT) if there exist positive numbers τ_{pq} such that $t_{p_l+1} - t_{p_l} \geq \tau_{pq}$ with $\sigma(t_{p_l}) = p$ and $\sigma(t_{p_l+1}) = q$, $\forall l \in \mathbb{N}^+$. Furthermore, we indicate the fact that the next mode to be switched on after p is q with $\mathcal{N}(p) = q$. The MMDDT switching law is defined for every p, q such that $\mathcal{N}(p) = q$.*

The control objective of the switched system in this chapter is presented as:

Problem 3.1 *Develop an adaptive mechanism and a switching law that, without requiring the knowledge of the actual values of A_p and B_p, $p \in \mathcal{M}$, assures the global uniform ultimate boundedness of all signals of the closed-loop switched system.*

3.3 Design of switched adaptive controllers

In this section, to solve *Problem 3.1*, an adaptive law and three switching laws, which are based on MDDT, MMDDT, and MDADT, are proposed. Since $A_{\mathrm{m}p}$ is a Hurwitz matrix, there exist a matrix $P_p > 0$ and a constant $\kappa_p > 0$ for every subsystem $p \in \mathcal{M}$ such that

$$A_{\mathrm{m}p}^T P_p + P_p A_{\mathrm{m}p} + \kappa_p P_p \leq 0 \tag{3.6}$$

where κ_p is called a stability margin of subsystem p. In addition, we define $\overline{\lambda}_p$ an upper bound and $\underline{\lambda}_p$ a lower bound of the eigenvalues of P_p, respectively. We define $\kappa_{\max} = \max_{p\in\mathcal{M}} \kappa_p$, $\alpha = \max_{p\in\mathcal{M}} \overline{\lambda}_p$, and $\beta = \min_{p\in\mathcal{M}} \underline{\lambda}_p$.

3.3.1 Switching laws via extended dwell time and extended average dwell time

Firstly, we introduce a switching law $\sigma(\cdot)$ based on the following MDDT:

$$\tau_p = \frac{1+\zeta}{\kappa_p} \ln \mu_p, \quad \forall p \in \mathcal{M} \tag{3.7}$$

where $\mu_p = \alpha / \underline{\lambda}_p$ and ζ is a user-defined positive constant.

Assuming that the next subsystem q to be switched to after subsystem p is known, the MDDT switching law in (3.7) is extended to an MMDDT switching law that satisfies

$$\tau_{pq} = \frac{1+\zeta}{\kappa_p} \ln \mu_{pq}, \quad \forall p, q \in \mathcal{M} \text{ with } p \neq q \tag{3.8}$$

where $\mu_{pq} = \overline{\lambda}_q / \underline{\lambda}_p$ and ζ is a user-defined positive constant. The MMDDT (3.8) switching law represents a larger class of switching signals than (3.7), for which GUUB stability of the closed-loop switched system can be guaranteed.

Finally, a switching law is proposed based on the MDADT strategy as follows:

$$\tau_{\mathrm{a}p} > \tau_{\mathrm{a}p}^* = \frac{1+\zeta}{\kappa_p} \ln \mu_p, \quad \forall p \in \mathcal{M} \tag{3.9}$$

where $\mu_p = \alpha/\lambda_{\min}(P_p)$ and ζ is a user-defined positive constant which will be clarified in the next section.

3.3.2 Adaptive laws

Before introducing the adaptive law, the following assumptions are made:

Assumption 3.1 *There exists a matrix $S_p \in \mathbb{R}^{m\times m}$ such that $M_p := L_p^* S_p = \left(L_p^* S_p\right)^T = S_p^T L_p^{*T} > 0, \forall p \in \mathcal{M}$.*

Assumption 3.2 *Upper and lower bounds of K_p^* and L_p^* are known, i.e., $K_p^* \in [\underline{K}_p, \overline{K}_p]$ and $L_p^* \in [\underline{L}_p, \overline{L}_p]$, $\forall p \in \mathcal{M}$. The upper and lower bounds are to be interpreted component-wise.*

Remark 3.1 *Assumption 3.1* is widely adopted in adaptive law of multiple-input systems, where it is used to ensure the boundedness of signals in closed-loop systems [88, 102, 104, 120]. As for *Assumption 3.2* , since the parameter estimates may deviate far away from the nominal value due to switches between different subsystems which might destabilize the closed-loop system, the knowledge of an upper bound and a lower bound of the parameters prevents this case from happening with a projection law [91, 115]. □

Therefore, using a Lyapunov method [89, 91], the following adaptive law is adopted:

$$\begin{aligned} \dot{K}_{\sigma(t)}^T(t) &= -S_{\sigma(t)}^T B_{\mathrm{m}\sigma(t)}^T P_{\sigma(t)} e(t) x^T(t) + F_{x\sigma(t)}^T(t) \\ \dot{L}_{\sigma(t)}(t) &= -S_{\sigma(t)}^T B_{\mathrm{m}\sigma(t)}^T P_{\sigma(t)} e(t) r^T(t) + F_{r\sigma(t)}(t) \end{aligned} \tag{3.10}$$

where $F_{xp}(\cdot)$ and $F_{rp}(\cdot)$, $\forall p \in \mathcal{M}$, are parameters projection laws as given in [118]: Let

$$\begin{aligned} K_p &= \left[k_{p1}, \ldots, k_{pm}\right], \qquad L_p = \left[l_{p1}, \ldots, l_{pm}\right] \\ F_{xp} &= \left[f_{xp1}, \ldots, f_{xpm}\right], \quad F_{rp} = \left[f_{rp1}, \ldots, f_{rpm}\right] \\ \Phi_{xp} &= \left(-S_p^T B_{\mathrm{m}p}^T P_p e x^T\right)^T = [\phi_{xp1}, \ldots, \phi_{xpm}] \\ \Phi_{rp} &= -S_p^T B_{\mathrm{m}p}^T P_p e r^T = [\phi_{rp1}, \ldots, \phi_{rpm}]. \end{aligned}$$

Then, we obtain the projection terms as follows, for $i \in \{1, \cdots, m\}$

$$\begin{aligned} f_{xpi}(t) &= \begin{cases} -\phi_{xpi}(t) & \text{if } k_{pi}(t) \le \underline{k}_{pi} \text{ \& } \phi_{xpi}(t) \le 0, \\ & \text{or if } k_{pi}(t) \ge \overline{k}_{pi} \text{ \& } \phi_{xpi}(t) \ge 0 \\ 0 & \text{otherwise} \end{cases} \\ f_{rpi}(t) &= \begin{cases} -\phi_{rpi}(t) & \text{if } k_{pi}(t) \le \underline{k}_{pi} \text{ \& } \phi_{rpi}(t) \le 0, \\ & \text{or if } k_{pi}(t) \ge \overline{k}_{pi} \text{ \& } \phi_{rpi}(t) \ge 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.11}$$

which can guarantee the parameter estimates bounded.

Remark 3.2 At the switch-in instant $t_l^{p_{\text{in}}}$ of subsystem p, $\forall p \in \mathcal{M}$, $l \in \mathbb{N}^+$ the initial conditions of (3.10) are taken from the estimates available at the previous switch-out instant of the same subsystem, i.e., $K_p(t_l^{p_{\text{in}}}) = K_p(t_{l+1}^{p_{\text{out}}})$, and $L_p(t_l^{p_{\text{in}}}) = L_p(t_{l+1}^{p_{\text{out}}})$. This gives rise to continuity of the parameter estimates. Fig. 3.1 gives a conceptual illustration of the evolution of parameter estimates, where the parameter estimates for subsystem p are updated when subsystem p is active and they are kept invariant otherwise. □

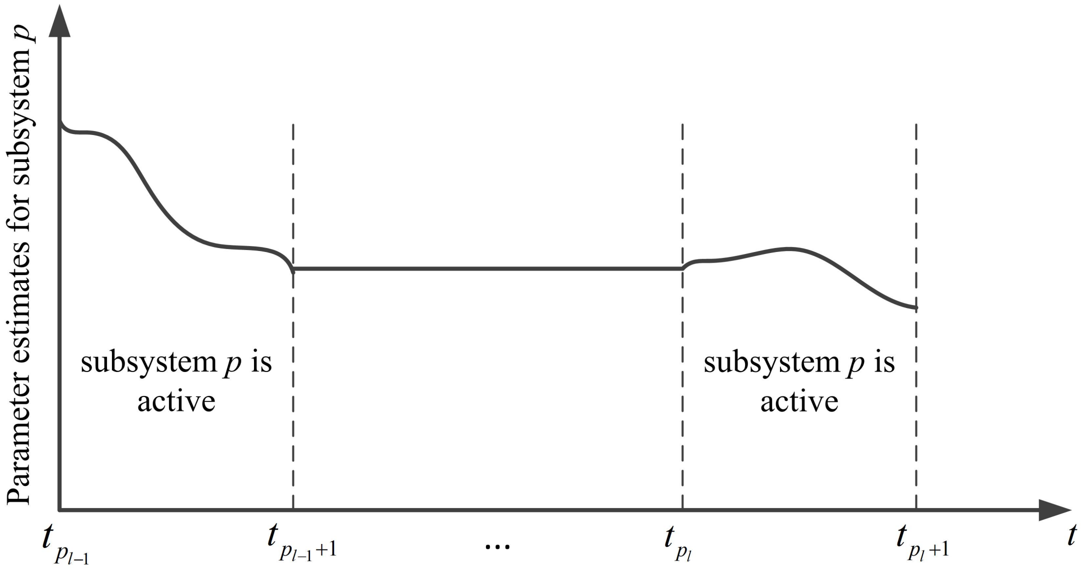

Figure 3.1: The evolution of parameter estimates.

3.4 Main results

3.4.1 Performance analysis with MDDT and MMDDT switching laws

Theorem 3.3 *With the adaptive law* (3.10) *and the switching law based on MDDT* (3.7), *the GUUB stability of the unknown switched systems* (3.1) *can be guaranteed. In addition, the tracking error is bounded as*

$$\|e(t)\|^2 \leq \max\left\{\frac{\alpha}{\beta}\|e(t_0)\|^2 + \frac{1}{\beta}\sum_{p=1}^{M}\Xi_p, \frac{\alpha(1+\zeta)}{\beta^2\zeta}\sum_{p=1}^{M}\Xi_p\right\}. \tag{3.12}$$

Furthermore, the tracking error is GUUB with an ultimate bound b_{T} in the interval,

$$\left[0, \sqrt{\frac{\alpha(1+\zeta)}{\beta^2\zeta}\sum_{p=1}^{M}\Xi_p}\right] \tag{3.13}$$

where

$$\Xi_p = \mathrm{tr}\left[\left(\overline{K}_p - \underline{K}_p\right)M_p^{-1}\left(\overline{K}_p - \underline{K}_p\right)^T\right] + \mathrm{tr}\left[\left(\overline{L}_p - \underline{L}_p\right)^T M_p^{-1}\left(\overline{L}_p - \underline{L}_p\right)\right]. \tag{3.14}$$

Proof: The proof is organized as follows: we propose a Lyapunov function considering the tracking error and estimation errors, whose behavior is studied with the proposed adaptive law (3.10) and switching law based on MDDT (3.7). It is shown that the Lyapunov function is decreasing during any switching interval between two consecutive switching instants excluding switching instants. Then, it is proven that there exists a finite bound such that after some time the Lyapunov function will stay below the bound, which implies that the closed-loop system is GUUB.

Firstly, the following Lyapunov function is adopted:

$$V(t)=e^T(t)P_{\sigma(t)}e(t)+\sum_{p=1}^{M}\operatorname{tr}\left[\tilde{K}_p(t)M_p^{-1}\tilde{K}_p^T(t)\right]+\sum_{p=1}^{M}\operatorname{tr}\left[\tilde{L}_p^T(t)M_p^{-1}\tilde{L}_p(t)\right]. \tag{3.15}$$

Generally, the matrix $P_{\sigma(t)}$ is different for different subsystems, which indicates that $V(t)$ might be continuous w.r.t. time only in the intervals between two consecutive switches. In light of this, to investigate the behavior of $V(t)$, firstly, we need to establish the characteristics of $V(t)$ at the switching instants. Without loss of generality, we consider the Lyapunov function at the switching instant t_{l+1}, $l\in\mathbb{N}^+$. Subsystem $\sigma(t_{l+1}^-)$ is active when $t\in[t_l,\ t_{l+1})$ and subsystem $\sigma(t_{l+1})$ is active when $t\in[t_{l+1},\ t_{l+2})$. At the switching instant t_{l+1}, we have before switching

$$V(t_{l+1}^-)=e^T(t_{l+1}^-)P_{\sigma(t_{l+1}^-)}e(t_{l+1}^-)+\sum_{p=1}^{M}\operatorname{tr}\left[\tilde{K}_p(t_{l+1}^-)M_p^{-1}\tilde{K}_p^T(t_{l+1}^-)\right]+\sum_{p=1}^{M}\operatorname{tr}\left[\tilde{L}_p^T(t_{l+1}^-)M_p^{-1}\tilde{L}_p(t_{l+1}^-)\right]$$

and after switching

$$V(t_{l+1})=e^T(t_{l+1})P_{t_{l+1}}e(t_{l+1})+\sum_{p=1}^{M}\operatorname{tr}\left[\tilde{K}_p(t_{l+1})M_p^{-1}\tilde{K}_p^T(t_{l+1})\right]+\sum_{p=1}^{M}\operatorname{tr}\left[\tilde{L}_p^T(t_{l+1})M_p^{-1}\tilde{L}_p(t_{l+1})\right].$$

Based on the continuity of the tracking error $e(\cdot)$ in (3.5) and the continuity of the parameter estimates in (3.10), we have $e(t_{l+1})=e(t_{l+1}^-)$, $\tilde{K}_p(t_{l+1})=\tilde{K}_p(t_{l+1}^-)$, and $\tilde{L}_p(t_{l+1})=\tilde{L}_p(t_{l+1}^-)$ for any switching law. Then, due to the fact that $e^T(t)P_{\sigma(t)}e(t)\le\alpha e^T(t)e(t)$ and $e^T(t)P_{\sigma(t)}e(t)\ge\underline{\lambda}_{\sigma(t)}e^T(t)e(t)$, the quantitative relationship of $V(t)$ at the switching instant t_{l+1} is established as follows,

$$\begin{aligned}V(t_{l+1})-V(t_{l+1}^-)&=e^T(t_{l+1})\left(P_{\sigma(t_{l+1})}-P_{\sigma(t_{l+1}^-)}\right)e(t_{l+1})\\&\le\frac{\alpha-\underline{\lambda}_{\sigma(t_{l+1}^-)}}{\underline{\lambda}_{\sigma(t_{l+1}^-)}}e^T(t_{l+1})P_{\sigma(t_{l+1}^-)}e(t_{l+1})\\&\le\frac{\alpha-\underline{\lambda}_{\sigma(t_{l+1}^-)}}{\underline{\lambda}_{\sigma(t_{l+1}^-)}}V(t_{l+1}^-)\end{aligned}$$

i.e.,

$$V(t_{l+1})\le\mu_{\sigma(t_{l+1}^-)}V(t_{l+1}^-) \tag{3.16}$$

with

$$\mu_{\sigma(t_{l+1}^-)}=\alpha/\underline{\lambda}_{\sigma(t_{l+1}^-)}.$$

Next, the behavior of $V(t)$ between two consecutive discontinuities is studied. When

$t \in [t_l,\ t_{l+1})$, the derivative of $V(t)$ w.r.t. time using (3.3) and (3.5)–(3.10) is

$$\begin{aligned}\dot V(t) &= e^T(t)(A^T_{\mathrm{m}\sigma(t^-_{l+1})}P_{\sigma(t^-_{l+1})} + P_{\sigma(t^-_{l+1})}A_{\mathrm{m}\sigma(t^-_{l+1})})e(t) + 2\,\mathrm{tr}\left[\tilde K_{\sigma(t^-_{l+1})}(t)M^{-1}_{\sigma(t^-_{l+1})}F^T_{x\sigma(t^-_{l+1})}(t)\right]\\ &\quad + 2\,\mathrm{tr}\left[\tilde L^T_{\sigma(t^-_{l+1})}(t)M^{-1}_{\sigma(t^-_{l+1})}F_{r\sigma(t^-_{l+1})}(t)\right]\\ &\le -\kappa_{\sigma(t^-_{l+1})}e^T(t)P_{\sigma(t^-_{l+1})}e(t) + 2\,\mathrm{tr}\left[\tilde K_{\sigma(t^-_{l+1})}(t)M^{-1}_{\sigma(t^-_{l+1})}F^T_{x\sigma(t^-_{l+1})}(t)\right]\\ &\quad + 2\,\mathrm{tr}\left[\tilde L^T_{\sigma(t^-_{l+1})}(t)M^{-1}_{\sigma(t^-_{l+1})}F_{r\sigma(t^-_{l+1})}(t)\right].\end{aligned} \tag{3.17}$$

Using the definition of parameter projection (3.11), we have the following arguments:

- when $k_{pi} \in (\underline{k}_{pi}, \overline{k}_{pi})$, and $l_{pi} \in (\underline{k}_{pi}, \overline{k}_{pi})$, we have $f_{xpi} = 0$ and $f_{rpi} = 0$, which indicates that $\mathrm{tr}(\tilde K_p M_p^{-1} F^T_{xp}) = 0$ and $\mathrm{tr}(\tilde L^T_p M_p^{-1} F_{rp}) = 0$;

- when $k_{pi} = \overline{k}_{pi}$ with $\phi_{xpi} \le 0$, and $l_{pi} = \overline{l}_{pi}$ with $\phi_{rpi} \le 0$, we have $f_{xpi} = 0$ and $f_{rpi} = 0$, which indicates that $\mathrm{tr}(\tilde K_p M_p^{-1} F^T_{xp}) = 0$ and $\mathrm{tr}(\tilde L^T_p M_p^{-1} F_{rp}) = 0$;

- when $k_{pi} = \underline{k}_{pi}$ with $\phi_{xpi} \ge 0$, and $l_{pi} = \underline{l}_{pi}$ with $\phi_{rpi} \ge 0$, we have $f_{xpi} = 0$ and $f_{rpi} = 0$, which indicates that $\mathrm{tr}(\tilde K_p M_p^{-1} F^T_{xp}) = 0$ and $\mathrm{tr}(\tilde L^T_p M_p^{-1} F_{rp}) = 0$;

- when $k_{pi} = \overline{k}_{pi}$ with $\phi_{xpi} \ge 0$, and $l_{pi} = \overline{l}_{pi}$ with $\phi_{rpi} \ge 0$, we have $f_{xpi} = -\phi_{xpi} \le 0$ and $f_{rpi} = \phi_{xpi} \le 0$, which indicates that $\mathrm{tr}(\tilde K_p M_p^{-1} F^T_{xp}) \le 0$ and $\mathrm{tr}(\tilde L^T_p M_p^{-1} F_{rp}) \le 0$ considering $\tilde k_{pi} = k_{pi} - k^*_{pi} \ge 0$ and $\tilde l_{pi} = l_{pi} - l^*_{pi} \ge 0$;

- when $k_{pi} = \underline{k}_{pi}$ with $\phi_{xpi} \le 0$, and $l_{pi} = \overline{l}_{pi}$ with $\phi_{xpi} \le 0$, we have $f_{xpi} = -\phi_{xpi} \ge 0$ and $f_{rpi} = -\phi_{rpi} \ge 0$, which indicates that $\mathrm{tr}(\tilde K_p M_p^{-1} F^T_{xp}) \le 0$ and $\mathrm{tr}(\tilde L^T_p M_p^{-1} F_{rp}) \le 0$ considering $\tilde k_{pi} = k_{pi} - k^*_{pi} \le 0$ and $\tilde l_{pi} = l_{pi} - l^*_{pi} \le 0$.

In view of these arguments, it follows that $\mathrm{tr}(\tilde K_p M_p^{-1} F^T_{xp}) \le 0$ and $\mathrm{tr}(\tilde L^T_p M_p^{-1} F_{rp}) \le 0$, and thus $\dot V(t) \le 0$ in (3.17). Since the parameter estimation errors $\tilde K_p$ and $\tilde L_p$ are bounded using parameter projection laws F_{xp} and F_{rp}, the summation terms of (3.15) are upper bounded by $\sum_{p=1}^M \Xi_p$. Hence, it follows from (3.14), (3.15), and (3.17) that when $t \in [t_l,\ t_{l+1})$, $\forall\zeta > 0$,

$$\begin{aligned}\dot V(t) &\le -\kappa_{\sigma(t^-_{l+1})}e^T(t)P_{\sigma(t^-_{l+1})}e(t)\\ &\le -\kappa_{\sigma(t^-_{l+1})}V(t) + \kappa_{\sigma(t^-_{l+1})}\sum_{p=1}^M \Xi_p\\ &\le -\frac{\kappa_{\sigma(t^-_{l+1})}}{1+\zeta}V(t) + \frac{\kappa_{\sigma(t^-_{l+1})}}{1+\zeta}\left[(1+\zeta)\sum_{p=1}^M \Xi_p - \zeta V(t)\right].\end{aligned} \tag{3.18}$$

According to (3.17), with the adaptive law (3.10), $V(\cdot)$ is decreasing in between two consecutive switching instants, i.e., in the time interval $[t_l,\ t_{l+1})$. To analyze the behavior of the Lyapunov function, two possible scenarios should be taken into account:

- Case 1: when $V(t) \ge \sum_{p=1}^M \Xi_p(1+\zeta)/\zeta$, it follows that $\dot V(t) \le -\kappa_p/(1+\zeta)V(t)$, i.e., $V(t)$ is decreasing at an exponential rate.

- Case 2: when $V(t) \le \sum_{p=1}^M \Xi_p(1+\zeta)/\zeta$, it follows that $\dot V(t) \le 0$, i.e., $V(t)$ is non-increasing.

The value of $V(t_0)$ can be larger than $\sum_{p=1}^{M}\Xi_p(1+\zeta)/\zeta$ (*case 1*), less or equal than $\sum_{p=1}^{M}\Xi_p(1+\zeta)/\zeta$ (*case 2*).

Case 1: we assume $V(t) > \sum_{p=1}^{M}\Xi_p(1+\zeta)/\zeta$ when $t\in[t_0,\ t_0+T_1)$, which implies that $V(t)$ is decreasing at an exponential rate in the beginning. Denote the number of intervals that subsystem p, $p\in\mathcal{M}$, is active for $t\in[t_0,\ t_0+T_1)$ by $N_{1p}(t)$, and the number of all intervals in the time interval $[t_0,\ t]$ for $t\in[t_0,\ t_0+T_1)$ by $N_1(t)$, where T_1 represents the length of the time interval before $V(t)$ enters into the bound $\sum_{p=1}^{M}\Xi_p(1+\zeta)/\zeta$ from t_0. Denote the number of all intervals in the whole time interval $[t_0,\ t_0+T_1)$ by N_1. Therefore, when $t\in[t_0,\ t_0+T_1)$, since $V(t)$ is decreasing at an exponential rate, it follows with (3.16) that

$$\begin{aligned}
V(t) &\le V(t_{N_1(t)})\\
&\le \mu_{\sigma(t^-_{N_1(t)-1})}\exp\left(-\frac{\kappa_{\sigma(t_{N_1(t)})}}{1+\zeta}\left(t_{N_1(t)}-t_{N_1(t)-1}\right)\right)V(t_{N_1(t)-1})\\
&\ \ \vdots\\
&\le \mu_{\sigma(t^-_{N_1(t)-1})}\exp\left(-\frac{\kappa_{\sigma(t_{N_1(t)})}}{1+\zeta}\left(t_{N_1(t)}-t_{N_1(t)-1}\right)\right)\\
&\quad \mu_{\sigma(t^-_{N_1(t)-2})}\exp\left(-\frac{\kappa_{\sigma(t_{N_1(t)-1})}}{1+\zeta}\left(t_{N_1(t)-1}-t_{N_1(t)-2}\right)\right)\cdots\\
&\quad \mu_{\sigma(t_0)}\exp\left(-\frac{\kappa_{\sigma(t_0)}}{1+\zeta}\left(t_1-t_0\right)\right)V(t_0)\\
&\le \prod_{p=1}^{M}\mu_p^{N_{1p}(t)}\exp\left\{-\sum_{p=1}^{M}N_{1p}(t)\frac{\kappa_p}{1+\zeta}(t_{l+1}^{p_{\text{out}}}-t_l^{p_{\text{in}}})\right\}V(t_0)
\end{aligned} \tag{3.19}$$

where $l=\{0,\ 1,\ 2,\ \cdots,\ N_{1p}(t)\}$.

Substituting the MDDT condition $\tau_p = t_{l+1}^{p_{\text{out}}}-t_l^{p_{\text{in}}} \ge (1+\zeta)/(2\kappa_p)\ln\mu_p$ into (3.17), we have

$$V(t)\le V(t_0),\quad t\in[t_0,\ t_0+T_1). \tag{3.20}$$

Moreover, it is apparent that $V(t)\le\sum_{p=1}^{M}\Xi_p(1+\zeta)/\zeta$ for $t\in[t_0+T_1,\ t_{N_1+1})$ considering that $V(\cdot)$ is non-increasing in the time interval $[t_0+T_1,\ t_{N_1+1})$. Therefore, at the switching instant t_{N_1+1}, we have

$$V(t_{N_1+1})\le\frac{\alpha(1+\zeta)}{\beta\zeta}\sum_{p=1}^{M}\Xi_p. \tag{3.21}$$

Similarly, assume when $t\in[t_{N_1+1},\ t_0+T_2)$, $V(\cdot)$ is decreasing at an exponential rate. Denote the number of all intervals for $t\in[t_{N_1+1},\ t_0+T_2)$ by N_2. Then, substituting $\sum_{p=1}^{M}\Xi_p\alpha(1+\zeta)/(\beta\zeta)$ for $V(t_0)$ in (3.19), we can use the deduction similar to (3.19)–(3.21), and it follows that

$$V(t)\le\frac{\alpha(1+\zeta)}{\beta\zeta}\sum_{p=1}^{M}\Xi_p,\quad t\in[t_{N_1+1},\ t_0+T_2). \tag{3.22}$$

Furthermore, when $t\in[t_0+T_2,\ t_{N_1+N_2+2})$, since $V(\cdot)$ is non-increasing, it holds that at switching instant $t_{N_1+N_2+2}$

$$V(t_{N_1+N_2+2})\le\frac{\alpha(1+\zeta)}{\beta\zeta}\sum_{p=1}^{N}\Xi_p. \tag{3.23}$$

For $t \geq t_{N_1+N_2+2}$, using the similar analysis as (3.19)–(3.21) recursively, it holds that

$$V(t) \leq \frac{\alpha(1+\zeta)}{\beta\zeta} \sum_{p=1}^{M} \Xi_p, \quad t \in [t_{N_1+N_2+2}, \infty). \tag{3.24}$$

Therefore, with (3.21)–(3.23), we have

$$V(t) \leq \frac{\alpha(1+\zeta)}{\beta\zeta} \sum_{p=1}^{M} \Xi_p, \quad t \in [t_0 + T_1, \infty) \tag{3.25}$$

which implies that once $V(\cdot)$ enters the interval $[0, \sum_{p=1}^{M} \Xi_p(1+\zeta)/\zeta]$, it cannot exceed the bound $\sum_{p=1}^{M} \Xi_p \alpha(1+\zeta)/(\beta\zeta)$ for any time later with MDDT (3.7). Therefore, the closed-loop switched system is GUUB with the adaptive law (3.10) and the switching law based on (3.7).

Finally, the dynamics of the tracking error is studied. It follows that via (3.25) and (3.21)

$$V(t) \leq \max\left\{ V(t_0), \frac{\alpha(1+\zeta)}{\beta\zeta} \sum_{p=1}^{M} \Xi_p \right\}, \quad \forall t > 0. \tag{3.26}$$

In addition, considering that $e^T(t)P_{\sigma(t)}e(t) \geq \beta\|e(t)\|^2$, it is clear that

$$\begin{aligned} V(t) &= e^T(t)P_{\sigma(t)}e(t) + \sum_{p=1}^{M} \mathrm{tr}\left[\tilde{K}_p(t)M_p^{-1}\tilde{K}_p^T(t)\right] + \sum_{p=1}^{M} \mathrm{tr}\left[\tilde{L}_p^T(t)M_p^{-1}\tilde{L}_p(t)\right] \\ &\geq \beta\|e(t)\|^2 \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} V(t_0) &= e^T(t_0)P_{\sigma(t_0)}e(t_0) + \sum_{p=1}^{M} \mathrm{tr}\left[\tilde{K}_p(t_0)M_p^{-1}\tilde{K}_p^T(t_0)\right] + \sum_{p=1}^{M} \mathrm{tr}\left[\tilde{L}_p^T(t_0)M_p^{-1}\tilde{L}_p(t_0)\right] \\ &\leq \alpha\|e(t_0)\|^2 + \sum_{p=1}^{M} \Xi_p. \end{aligned} \tag{3.28}$$

Then, it follows from (3.25)–(3.28) that

$$\|e(t)\|^2 \leq \max\left\{ \frac{\alpha}{\beta}\|e(t_0)\|^2 + \frac{1}{\beta}\sum_{p=1}^{M} \Xi_p, \ \frac{\alpha(1+\zeta)}{\beta^2\zeta} \sum_{p=1}^{M} \Xi_p \right\}. \tag{3.29}$$

Furthermore, according to (3.23) and (3.28), the tracking error is GUUB with an ultimate bound b as in (3.13).

Case 2: we assume $V(t_0) \leq \sum_{p=1}^{M} \Xi_p(1+\zeta)/\zeta$, which implies that $V(\cdot)$ is non-increasing in the beginning. The same results (GUUB stability of the closed-loop switched system, (3.12) and (3.13)) can be obtained following the proof lines from (3.21) to (3.28). This completes the proof. □

Remark 3.4 If there exists a common positive definite matrix P satisfying LMI (3.6) for all $A_{\mathrm{m}p}$, the Lyapunov function is strictly decreasing for any $e(t) \neq 0$ and for any switching signal $\sigma(\cdot)$, and the tracking error tends to zero asymptotically [91]. In this case the ultimate bound b_{T} becomes zero, and global uniform ultimate stability becomes global asymptotic stability. □

Remark 3.5 The positive constant ζ shows the trade-off between the requirements on the switching signal and the behavior of the tracking error. When a faster switching rate is demanded, a smaller ζ should be selected according to (3.7) and (3.8), which results in a larger upper bound and ultimate bound of the tracking error according to (3.12) and (3.13). Vice versa, when a smaller upper bound and ultimate bound of the tracking error is demanded, a larger ζ should be selected, which results in a slower switching rate. □

Remark 3.6 The upper and ultimate bounds of the tracking error indicate that the proposed methods prevent the tracking error in the closed-loop switched system from growing large over short time intervals. In contrast, large signals may be observed in ADT and MDADT algorithms when the interval between two consecutive switches is very small [22]. □

For the case when the switching sequence is known, the following result is introduced.

Theorem 3.7 *With the control law* (3.4)*, the adaptive law* (3.10)*, and the switching law based on MMDDT* (3.8)*, the GUUB stability of the unknown switched system* (3.1) *can be guaranteed. In addition, the tracking error is bounded as:*

$$\|e(t)\|^2 \le \max\left\{\frac{\alpha}{\beta}\|e(t_0)\|^2 + \frac{1}{\beta}\sum_{p=1}^{M}\Xi_p,\ \max_{\substack{p,\,q\in\mathcal{M}\\ \mathcal{N}(p)=q}}\{\mu_{pq}\}\frac{(1+\zeta)}{\beta\zeta}\sum_{p=1}^{M}\Xi_p\right\} \tag{3.30}$$

where c_1 and c_2 are the same positive constants as in Theorem 3.3. In addition, the tracking error is GUUB with an ultimate bound b_{T} in the interval

$$\left[0,\ \sqrt{\max_{\substack{p,\,q\in\mathcal{M}\\ \mathcal{N}(p)=q}}\{\mu_{pq}\}\frac{(1+\zeta)}{\beta\zeta}\sum_{p=1}^{M}\Xi_p}\right]. \tag{3.31}$$

Proof: The proof is similar with the proof of Theorem 3.3. The same Lyapunov function as (3.15) is adopted. The main difference arises from the relationship of the values between the Lyapunov function at switching instant t_{l+1}, which is expressed as follows:

$$V(t_{l+1}) \le \frac{\overline{\lambda}_{\sigma(t_{l+1})}}{\underline{\lambda}_{\sigma(t_{l+1}^-)}}V(t_{l+1}^-) =: \mu_{\sigma(t_{l+1})\sigma(t_{l+1}^-)}V(t_{l+1}^-).$$

The dynamics of the Lyapunov function during the switching interval is identical with (3.17)–(3.17). Since the switching sequence is known, the maximum increase of the Lyapunov function at the switching instants is $\max_{p,\,q\in\mathcal{M},\mathcal{N}(p)=q}\{\mu_{pq}\}$ instead of α/β as in the MDDT case. The rest of the proof follows the lines from (3.16) to (3.30) after substituting $\mu_{\sigma(t_{l+1}^-)}$ with $\mu_{\sigma(t_{l+1})\sigma(t_{l+1}^-)}$. We conclude that the adaptive law (3.10) and the switching law with MMDDT (3.8) lead to GUUB stability with bounds (3.30) and (3.31). □

Remark 3.8 The GUUB stability of the closed-loop switched systems based on the proposed switching laws can be explained in a more intuitive way. The Lyapunov function (3.15) can be seen as the total energy of the uncertain switched system (3.1) due to the tracking error and parameter estimation errors. According to (3.18), during each interval between two consecutive switches, the energy is decreasing. However, when the switching takes place, some energy might be added to the switched system. In light of this, what we have done with the MDDT and MMDDT schemes is actually eliminating the incremental energy at each switching instant by making the energy-decreasing interval long enough. When the energy decreases to an extent that the reducing energy and incremental energy is keeping balance, the tracking error is thus kept in a certain interval later. □

The following corollary to *Theorem 3.10* can be established.

Corollary 3.1 *Consider two consecutive switching instants t_l and t_{l+1}, $l \in \mathbb{N}^+$, with $\sigma(t_l) = p$ and $\sigma(t_{l+1}) = q$, $p, q \in \mathcal{M} \times \mathcal{M}$. If $\mu_{pq} \leq 1$ in* (3.8)*, then the switching interval $t_{l+1} - t_l$ can be as small as desired, i.e., the closed-loop switched system is GUUB and the tracking error is upper bounded as* (3.12) *with ultimate bound b_{T} as* (3.13) *under arbitrarily fast switches of the subsystem p when $\mathcal{N}(p) = q$, where $\mathcal{N}(p)$ denotes the index of subsystem to be switched on after subsystem p.*

Proof: Since $\mu_{pq} \leq 1$ in (3.8), it follows that $V(t_{l+1}) \leq V(t_{l+1}^-)$ at the switching instant t_{l+1}, which indicates the energy defined by the Lyapunov function is decreasing at the switching instant t_{l+1}. Considering that the Lyapunov function is non-increasing in the interval between two consecutive switching instants, τ_{pq} is allowed to be arbitrarily small. Therefore, the closed-loop systems are GUUB with arbitrarily fast switches of the subsystem p when $\mathcal{N}(p) = q$. □

Remark 3.9 Note that Corollary 3.1 does not guarantee asymptotic stability under arbitrarily fast switches, unless a common Lyapunov function exists as discussed in Remark 3.4. For example, consider two subsystems p and q, for which the condition $\mu_{pq} \leq 1$ is satisfied: the system can switch arbitrarily fast from p to q, but if the switching signal at switching instant t_{p_l+1} switches from q to p, we have $\mu_{qp} \geq 1$, which leads to GUUB stability since the Lyapunov function may increase at switching instant t_{p_l+1}. □

3.4.2 Performance analysis with MDADT switching laws

Theorem 3.10 *With the control law* (3.4)*, the adaptive law* (3.10) *and the switching law based on MDADT* (3.9)*, the GUUB stability of the uncertain switched system* (3.1) *can be guaranteed. The norm of the tracking error is upper bounded by, $\forall\, t \geq t_0$,*

$$\|e(t)\|^2 \leq \frac{1}{\beta} \exp\left(\sum_{p=1}^{M} N_{0p} \ln \mu_p\right) \max\left\{ c_1,\ \frac{\alpha c_2 (1+\zeta)}{\beta \min_{p\in\mathcal{M}} \{\kappa_p\} \zeta} \right\} \tag{3.32}$$

where the positive constants c_1 and c_2 depend on the initial estimates and the real values of the controller parameters. In addition, the ultimate bound b_{T} for the tracking error lies in the interval

$$\left[0,\ \sqrt{\frac{\alpha(1+\zeta)}{\beta^2 \zeta} \sum_{p=1}^{M} \Xi_p}\,\right]. \tag{3.33}$$

Proof: The proof can be carried out in the similar vein of the one of Theorem 3.3 and thus is omitted. □

Remark 3.11 The positive number ζ can illustrate the trade-off between the length of the switching intervals of MDADT according to (3.9) and the performance of the tracking error according to (3.32) and (3.33). A large ζ results in a long MDADT, which means that the switching signal is more conservative and we have a smaller bound on the ultimate tracking error. On the contrary, a small ζ leads to a short MDADT and a larger ultimate bound of the tracking error. Finally, when the reference signal is zero, the tracking error turns out to be a regulation error, and the adaptive laws in (3.10) can still guarantee (3.32) and (3.33). □

Remark 3.12 The upper bound of the tracking error is dependent on the chatter bounds of MDADT N_{0p}, $p \in \mathcal{M}$: the smaller N_{0p} the better the transient performance of the tracking error. When $N_{0p} = 1$, $\forall p \in \mathcal{M}$, MDADT switching becomes mode-dependent dwell time switching [22]. Note that the upper bound of the norm of the tracking error is based on the worst-case scenario when the switches given by the chatters bounds N_{0p} occur right after t_0 and the intervals between two consecutive chatters are close to zero. □

Corollary 3.2 *If there exists a common Lyapunov function for all the subsystems of* (3.2), *i.e., there exists a positive definite matrix P such that $A_{\mathrm{m}p}^T P + PA_{\mathrm{m}p} + \kappa P \leq 0$, $\forall p \in \mathcal{M}$, asymptotic stability of the tracking error can be guaranteed using the adaptive law* (3.10) *with no leakage terms under arbitrarily fast switching.*

Proof: The proof follows the same lines of [91]. □

3.5 Example

In this section, a highly maneuverable aircraft technology vehicle [35, 157] is adopted to illustrate the proposed adaptive control method. The adaptive control approach is utilized to design closed-loop controllers and switching signals for the unstable longitudinal dynamics. Consider a switched linear system with the following three modes:

$$A_1 = \begin{bmatrix} -0.8435 & 0.97505 & -0.0048 \\ 8.7072 & -1.1643 & 0.0026 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1299 & -0.092 & -0.0107 & -0.0827 \\ -7.6833 & -4.7974 & 4.8178 & -5.7416 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1.8997 & 0.98312 & -0.00073 \\ 11.720 & -2.6316 & 0.00088 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.2436 & -0.1708 & -0.00497 & -0.1997 \\ -46.206 & -31.604 & 22.396 & -31.179 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.2206 & 0.99411 & -0.00084 \\ -64.071 & -1.8876 & 0.00046 \\ 0 & 1 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -0.0662 & -0.0315 & -0.0141 & -0.0749 \\ -27.333 & -13.163 & 11.058 & -26.878 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A. Design of reference model

Three LQR controllers $u = K_p^* x$ with $Q = \mathrm{diag}([1\ 1\ 5])$, $R = \mathrm{diag}([1\ 1\ 1\ 1])$ are adopted to design the reference model, i.e., $\dot{x}_{\mathrm{m}} = A_{\mathrm{m}p} x_{\mathrm{m}} + B_{\mathrm{m}p} r = (A_p + B_p K_p^*) x_{\mathrm{m}} + B_p r$, $p \in \mathcal{M}$. The nominal parameters and the system matrices of reference model are:

$$K_1^* = \begin{bmatrix} 0.6219 & 0.7469 & 1.4508 \\ 0.3969 & 0.4671 & 0.9013 \\ -0.3174 & -0.4621 & -0.9483 \\ 0.4534 & 0.5572 & 1.0902 \end{bmatrix}, \quad L_1^* = I_4, \ A_{\mathrm{m}1} = \begin{bmatrix} -0.9949 & 0.7939 & -0.3562 \\ -2.1076 & -14.5691 & -26.2966 \\ 0 & 1 & 0 \end{bmatrix}$$

$$K_2^* = \begin{bmatrix} 0.1984 & 0.6793 & 1.5202 \\ 0.1368 & 0.4646 & 1.0392 \\ -0.0642 & -0.3289 & -0.7527 \\ 0.1431 & 0.4585 & 1.0212 \end{bmatrix}, \quad L_2^* = I_4, \ A_{\mathrm{m}2} = \begin{bmatrix} -1.9997 & 0.6484 & -0.7487 \\ -7.6710 & -70.3615 & -151.7803 \\ 0 & 1 & 0 \end{bmatrix}$$

$$K_3^* = \begin{bmatrix} -0.6674 & 0.6397 & 1.4517 \\ -0.3220 & 0.3081 & 0.6995 \\ 0.3287 & -0.2599 & -0.6292 \\ -0.6423 & 0.6288 & 1.4175 \end{bmatrix}, \quad L_3^* = I_4, \ A_{\mathrm{m}3} = \begin{bmatrix} -1.1228 & 0.8986 & -0.2163 \\ -20.6916 & -43.2036 & -93.9421 \\ 0 & 1 & 0 \end{bmatrix}.$$

B. Adaptive control design

Let $\kappa_1 = 0.25, \kappa_2 = 0.5, \kappa_3 = 0.4$. Solving (3.6) gives rise to the following positive definite matrices:

$$P_1 = \begin{bmatrix} 0.7337 & -0.0162 & -0.3781 \\ -0.0162 & 0.0549 & 0.0800 \\ -0.3781 & 0.0800 & 2.3960 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5225 & -0.0028 & -0.0517 \\ -0.0028 & 0.0092 & 0.0132 \\ -0.0517 & 0.0132 & 1.9764 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.7942 & -0.0063 & -0.3177 \\ -0.0063 & 0.0167 & 0.0241 \\ -0.3177 & 0.0241 & 2.4767 \end{bmatrix}.$$

Then, the bounds of DT, MDDT, MMDDT are obtained as shown in Table 3.1, which shows that a bigger class of switching signals based on MDDT is obtained than the class of switching signals based on DT. Moreover, when the switching sequence is known, MMDDT leads to even less conservative switching signals than MDDT and DT.

Table 3.1: Comparison of three switching laws.

Switching laws	**DT**	**MDDT**	**MMDDT**
Switching sequences	Unknown	Unknown	Known in advance
Switching signals	$\tau_{\mathrm{d}}^* = 23.7$ $\mu = 278.3$ $\kappa = 0.25$	$\tau_1^* = 16.3$, $\tau_2^* = 11.8$, $\tau_3^* = 13.2$, $\mu_1 = 48.6$ $\mu_2 = 278.3$, $\mu_3 = 154.1$ $\kappa_1 = 0.25$, $\kappa_2 = 0.5$, $\kappa_3 = 0.4$	$\tau_{13}^* = 16.3$, $\tau_{32}^* = 12.6$ $\tau_{21}^* = 11.8$, $\tau_{23}^* = 10$ $\mu_{13} = 48.6$, $\mu_{32} = 120.3$ $\mu_{21} = 272.4$ $\kappa_1 = 0.25$ $\kappa_2 = 0.5$, $\kappa_3 = 0.4$

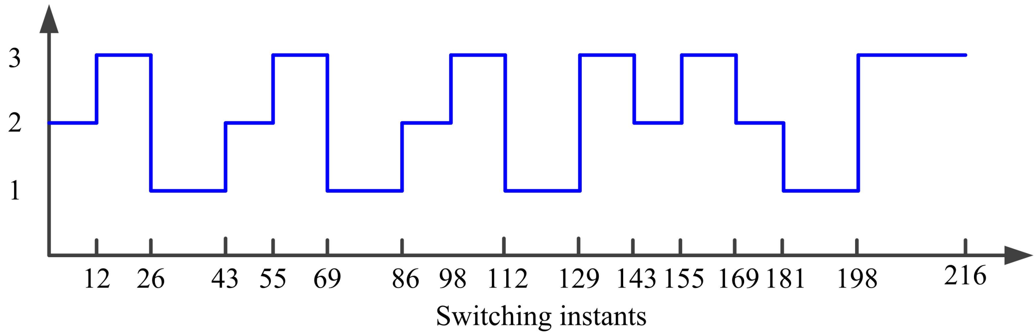

Figure 3.2: Switching signal based on MDDT.

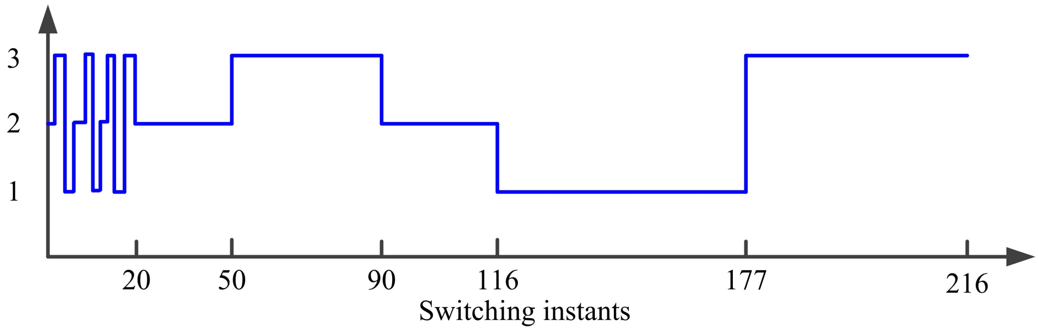

Figure 3.3: Switching signal based on MDADT.

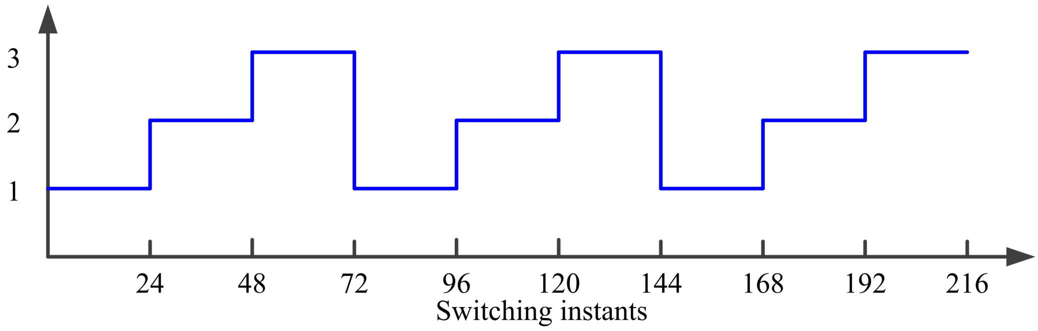

Figure 3.4: Switching signal based on DT.

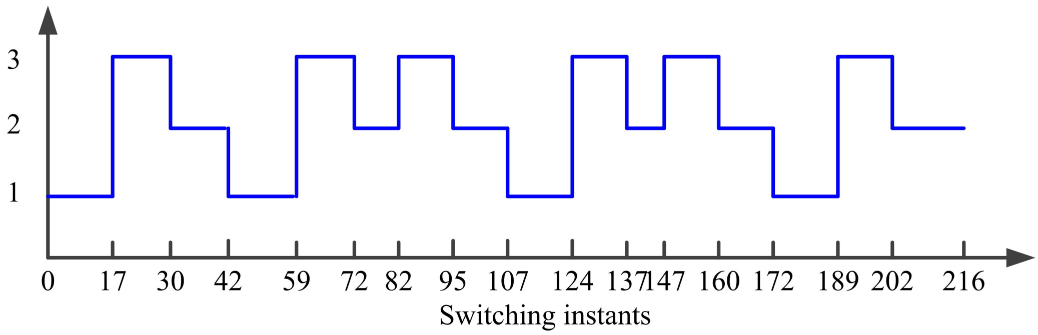

Figure 3.5: Switching signal based on MMDDT.

We design switching signals based on DT, MDADT, MDDT, and MMDDT as shown in Figs. 3.2–3.5, respectively. Consider the adaptive gains $S_1 = S_2 = S_3 = 10I_{4\times4}$, the initial conditions $x(0) = [0\ 0\ 0]^T$, $x_{\mathrm{m}}(0) = [2\ 2\ 1]^T$, $K_p(0) = 0.8K_p^*$, $L_p(0) = 0.8L_p^*$, and the refer-

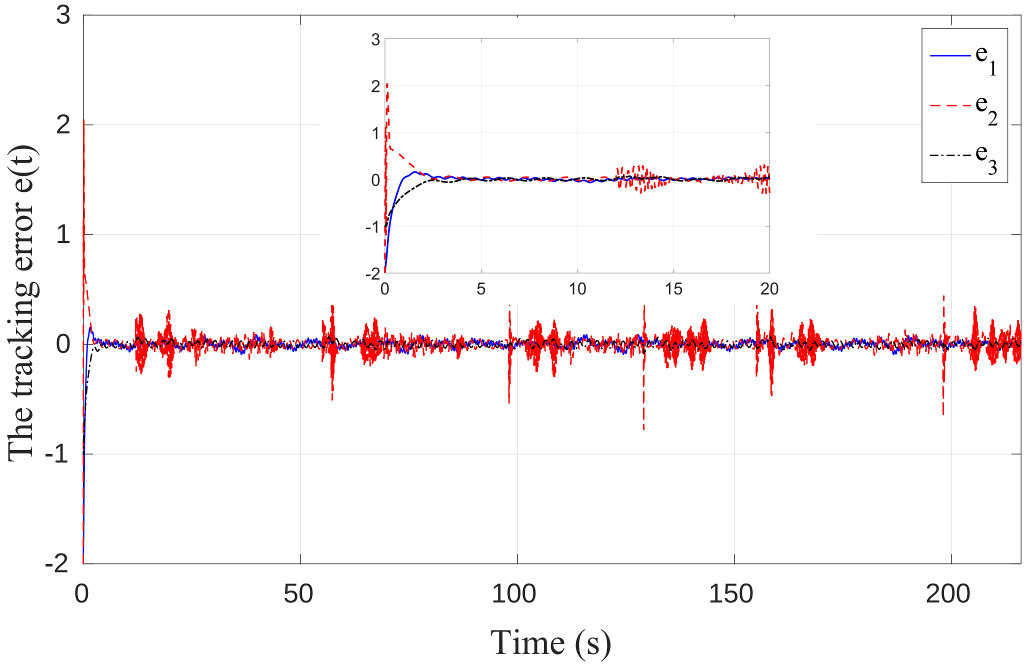

Figure 3.6: The tracking error based on MDDT with an enlarged detail in the time interval [0,20].

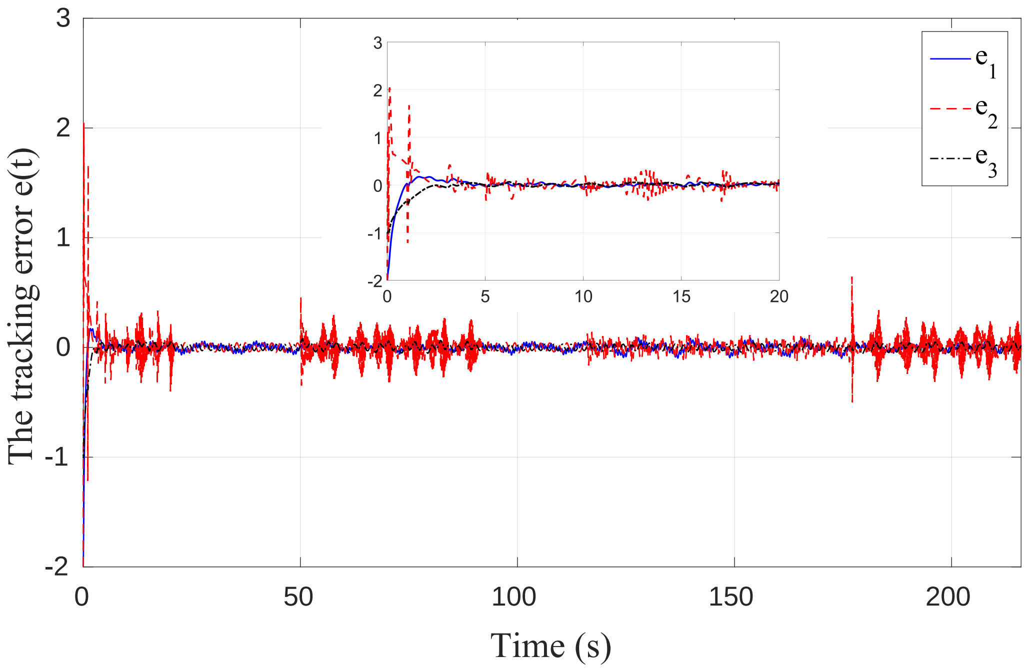

Figure 3.7: The tracking error based on MDADT with an enlarged detail in the time interval [0,20].

ence input $r(t) = [2\sin(t)\ \ \cos(t)\ \ 0.5\sin(0.5t)\ \ 0]^T$. Furthermore, to illustrate robustness of the adaptive law with parameter projection (3.10) against some external noises (the systematic design of robust adaptive laws will be introduced in Chapter 5), we add disturbances $d(t) = [0.2\sin(10t)\ 0.15e^{-t}\ 0.1\cos(\pi t)]^T$ to the switched linear systems. The tracking errors

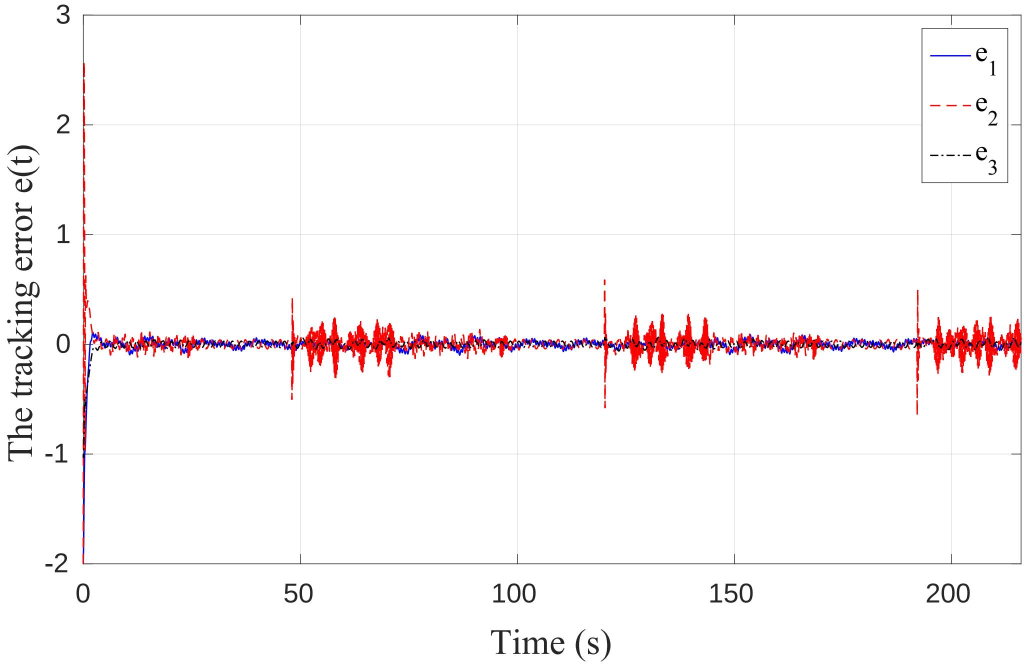

Figure 3.8: The tracking error based on DT.

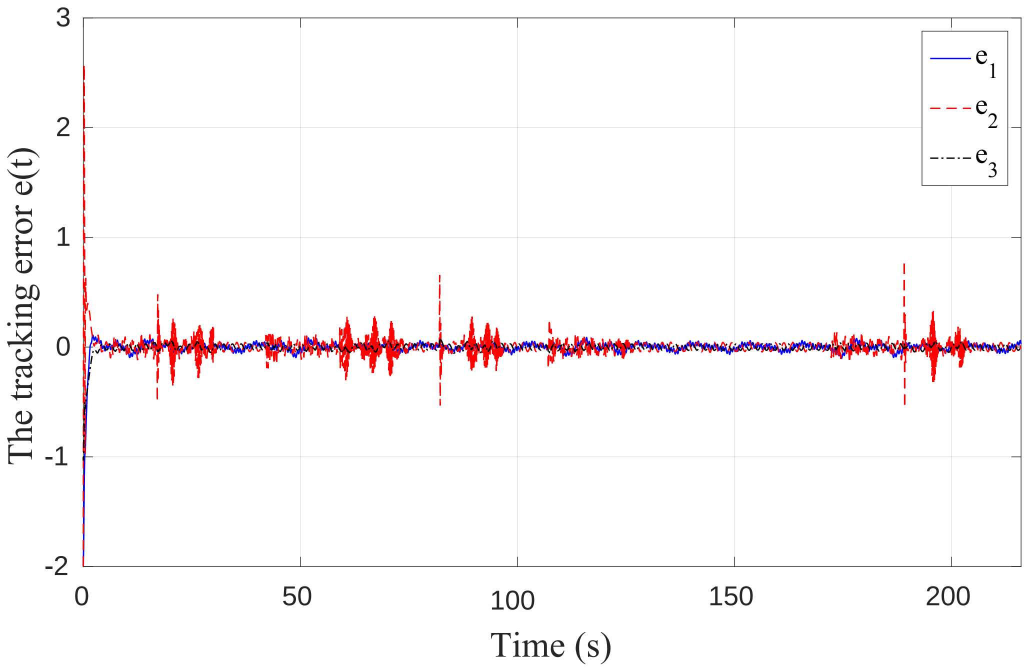

Figure 3.9: The tracking error based on MMDDT.

based on the four switching signals are shown in Figs. 3.6–3.9, respectively. It can be observed that the tracking errors are upper bounded and ultimately bounded, which is verified by the results of Theorems 3.3 and 3.10 in this chapter. Moreover, comparing Fig. 3.6 and Fig. 3.7, the fast switchings of MDADT negatively impact the transient performance of the tracking error (see the enlarged details).

3.6 Concluding remarks

In this chapter, an adaptive tracking control problem of uncertain switched linear systems has been studied. Switching laws by exploiting mode-dependent dwell time, mode-mode-dependent dwell time, and mode-dependent average dwell time have been developed, which are less conservative than ones based on the dwell time. Global uniform ultimate boundedness of the closed-loop switched system based on the proposed methods can be guaranteed. Moreover, a computable upper bound and an ultimate bound of the tracking error have been derived. Finally, an example of highly maneuverable aircraft technology has demonstrated the effectiveness of the proposed adaptive tracking control methods.

Chapter 4

Adaptive Asymptotic Tracking of Switched Linear Systems with Dwell Time

This chapter establishes a novel result about adaptive asymptotic tracking control of uncertain switched linear systems. The result exploits a recently proposed stability condition for switched systems. In particular, a Lyapunov function with a time-varying positive definite matrix is used to develop a novel piecewise continuous model-reference adaptive law and a dwell time switching law. In contrast with previous research, where asymptotic tracking was possible only in the presence of a common Lyapunov function for the reference models, in this chapter asymptotic tracking is shown in a more general setting. Additionally, in the presence of persistence of excitation, the controller parameter estimation errors will converge to zero asymptotically. The main contribution of this chapter consists in establishing a symmetry between adaptive control of classical non-switched linear systems and adaptive control of switched linear systems.

The research presented in this chapter has been published in [135].

4.1 Introduction

The results in literature [89–92, 117, 118] and their extended results presented in Chapter 3 fail to guarantee asymptotic stability of the tracking error in a general setting, i.e., when no common Lyapunov function exists for all subsystems, and to guarantee convergence of the parameter estimation errors to zero. This leads to a big theoretical gap between adaptive control of uncertain non-switched linear systems and adaptive control of uncertain switched linear systems. This chapter aims at filling this gap. In other words, we want to develop an adaptive law and a switching law for uncertain time-driven switched systems to achieve the same asymptotic stability results as the ones of adaptive control of classical non-switched systems. Furthermore, in presence of a persistently exciting reference input, we want to guarantee convergence of the controller parameter estimation errors to zero.

Recently, a new asymptotic stability condition for switched linear systems has been proposed based on a dwell-time switching law [1]. There are some distinguishing properties of this new stability condition with respect to those proposed in [19, 33, 140]. In particular, the dwell time guaranteeing the asymptotic stability can be calculated without involving an exponential term. Moreover, instead of a single positive definite matrix, a family of positive definite matrices is associated to each subsystem, which can be used to construct a time-varying positive definite matrix using the linear interpolation method for a quadratic Lyapunov function. The resulting Lyapunov function is decreasing during the intervals between two consecutive switching instants and non-increasing at the switching instants. In light of this, the current chapter exploits the aforementioned stability result to develop a novel model reference adaptive law for uncertain switched linear systems to guarantee asymptotic stability.

The main contributions of the chapter can be summarized as follows: first, in contrast with the previous chapter, the proposed adaptive laws completely remove the exponential decrease/bounded increase requirements of the Lyapunov function; second, there is no need for parameter projection and for the *a priori* knowledge of upper and lower bounds for the parameters when the switched system is not subject to disturbances; finally, asymptotic stability is established for the first time, i.e., the tracking error converges to zero asymptotically, even when no common Lyapunov function for the reference models exists. Furthermore, if the reference input is persistently exciting, we can also guarantee that the parameter estimates of the state feedback controller converge to the nominal parameters asymptotically, which makes the closed-loop switched system behave like the reference model. In view of these achievements, a symmetry between adaptive control of switched linear systems and adaptive control of non-switched systems is established.

The chapter is organized as follows: Section 4.2 presents the control problem and some preliminaries for later analysis. Section 4.3 proposes an adaptive law and a switching law to solve the adaptive asymptotic tracking problem. Stability results about closed-loop switched systems based on a quadratic Lyapunov function are presented Section 4.4. Section 4.5 adopts a practical example to illustrate the proposed results. Some concluding remarks are given in Section 4.6.

4.2 Problem statement

This chapter focuses on uncertain switched single-input linear systems[1] described by the following differential equation:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t), \quad \sigma(t) \in \mathcal{M} = \{1, \dots, M\} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state vector, and $u \in \mathbb{R}$ represents some piecewise continuous input. The matrices $A_p \in \mathbb{R}^{n\times n}$ and vectors $b_p \in \mathbb{R}^n$ are assumed to be unknown for all $p \in \mathcal{M}$. The switching signal $\sigma : [0,\infty) \to \mathcal{M} := \{1,2,\dots,M\}$ is a piecewise function with M denoting the number of subsystems. To develop the adaptive tracking scheme, a reference switched system representing the desired behavior of (4.1) is given as follows:

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}\sigma(t)}x_{\mathrm{m}}(t) + b_{\mathrm{m}\sigma(t)}r(t), \quad \sigma(t) \in \mathcal{M} \tag{4.2}$$

where $x_{\mathrm{m}} \in \mathbb{R}^n$ is the desired state vector, and $r \in \mathbb{R}$ is a bounded reference input. The matrices $A_{\mathrm{m}p} \in \mathbb{R}^{n\times n}$ and vectors $b_{\mathrm{m}p} \in \mathbb{R}^n$ are known, and $A_{\mathrm{m}p}$ are Hurwitz matrices for all $p \in \mathcal{M}$. Suppose that $(A_{\mathrm{m}p},\ b_{\mathrm{m}p})$ is controllable for each $p \in \mathcal{M}$ and each subsystem in (4.1) has its own corresponding reference submodel. We assume the measurement of $x(t)$ is available. Hence, the nominal state feedback controller that makes the switched system behave like

[1]Extension to multiple-input systems can be achieved as described in Chapter 3.

the reference model is given as follows:

$$u^*(t) = k^{*T}_{\sigma(t)}(t)x(t) + l^*_{\sigma(t)}(t)r(t)$$

where the nominal parameters $k^*_p \in \mathbb{R}^n$ and $l^*_p \in \mathbb{R}$ exist under the assumption that the following matching conditions hold [43, 51, 91]:

$$A_p + b_p k^{*T}_p = A_{\mathrm{m}p},\ b_p l^*_p = b_{\mathrm{m}p}. \tag{4.3}$$

However, since A_p and b_p are unknown, we cannot obtain k^*_p and l^*_p from (4.3). In light of this, the state feedback controller is developed as:

$$u(t) = k^T_{\sigma(t)}(t)x(t) + l_{\sigma(t)}(t)r(t) \tag{4.4}$$

where k_p and l_p are the estimates of k^*_p and l^*_p, respectively. In addition, we define the tracking error as: $e(t) = x(t) - x_{\mathrm{m}}(t)$.

Substituting (4.4) into (4.1), and subtracting (4.2), the dynamics of the tracking error are as follows:

$$\dot{e}(t) = A_{\mathrm{m}\sigma(t)}e(t) + b_{\sigma(t)}(\tilde{k}^T_{\sigma(t)}(t)x(t) + \tilde{l}_{\sigma(t)}(t)r(t)) \tag{4.5}$$

where $\tilde{k}_p = k_p - k^*_p$ and $\tilde{l}_p = l_p - l^*_p$ are the parameter estimation errors.

The problem addressed in this chapter is given as follows:

Problem 4.1 *Develop an adaptive law for k_p and l_p in* (4.4) *and a switching law $\sigma(\cdot)$ such that the switched system* (4.1) *with the state-feedback controller* (4.1) *can asymptotically track the reference switched system* (4.2)*, i.e., the tracking error satifies $e(t) \to 0$ as $t \to \infty$. In addition, convergence of the parameter estimates to nominal parameters is achieved if the reference input $r(\cdot)$ is persistently exciting, i.e., $\tilde{k}_p \to 0$ and $\tilde{l}_p \to 0$ as $t \to \infty$.*

Before presenting the main results, we assume that the sign of l^*_p is known, $\forall p \in \mathcal{M}$, which is a widely used assumption in adaptive control problems to ensure the boundedness of signals in closed-loop systems [43].

4.3 Design of switching laws and adaptive laws

To guarantee that the states x of the uncertain switched system track x_{m} asymptotically, firstly, we need to develop a dwell time switching law $\sigma(\cdot)$ to guarantee the global stability of the reference switched system with a bounded reference input r. It has been established that the globally asymptotic stability of the homogeneous system, $\dot{x}_{\mathrm{m}} = A_{\mathrm{m}p}x_{\mathrm{m}}$, $p \in \mathcal{M}$, is sufficient to lead to global stability of (4.2) [60]. Hence, using the stability condition proposed in [1], the following lemma is stated:

Lemma 4.1 *The switched system $\dot{x}_{\mathrm{m}} = A_{\mathrm{m}p}x_{\mathrm{m}}$, $p \in \mathcal{M}$, is globally asymptotically stable for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ if there exist: a collection of symmetric matrices $P_{p,k} \in \mathbb{R}^{n\times n}$, $p \in \mathcal{M}$, $k = 0,\dots,K$, and a sequence $\{\delta_k\}_{k=1}^K > 0$ with $\sum_{k=1}^K \delta_k = \tau_{\mathrm{d}}$ such that the following hold:*

$$P_{p,k} > 0 \tag{4.6a}$$

$$(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k}A_{\mathrm{m}p} + A^T_{\mathrm{m}p}P_{p,k} < 0 \tag{4.6b}$$

$$(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k+1}A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,k+1} < 0 \tag{4.6c}$$
$$\textit{for } k = 0,\dots,K-1$$
$$P_{p,K}A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,K} < 0 \tag{4.6d}$$
$$P_{p,K} - P_{q,0} \geq 0 \tag{4.6e}$$
$$\textit{for } q = 1,\dots,p-1,p+1,\dots M$$

where K is an integer that may be chosen a priori, according to the allowed computational complexity.

By solving the LMIs in (4.6), a collection of symmetric matrices $P_{p,k}$ and a dwell time τ_d can be obtained that will be utilized to develop a new adaptive law. Let $\mathcal{S}$ denotes the switching instants $\{t_i\}_{i\in\mathbb{N}}$ and define a time sequence $\{t_{i,k}\}_{k=0}^{K}$ with $t_{i,k+1} - t_{i,k} = \delta_{k+1}$, $k = 0,\dots, K-1$. Note that $t_{i,0} = t_i$ and $t_{i,K} - t_{i,0} = \tau_\mathrm{d}$, as shown in Fig. 4.1 .

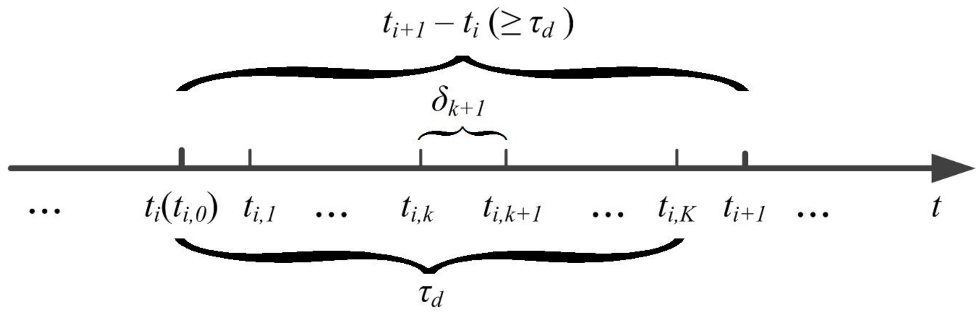

Figure 4.1: The time sequence between two consecutive switching instants.

Therefore, the adaptive law is proposed as follows:

$$\begin{aligned}\dot{k}_{\sigma(t)}(t) &= -\operatorname{sgn}[l^*_{\sigma(t)}]\Gamma_{\sigma(t)}x(t)e^T(t)P_{\sigma(t)}(t)b_{\mathrm{m}\sigma(t)}\\ \dot{l}_{\sigma(t)}(t) &= -\operatorname{sgn}[l^*_{\sigma(t)}]\gamma_{\sigma(t)}r(t)e^T(t)P_{\sigma(t)}(t)b_{\mathrm{m}\sigma(t)}\end{aligned} \tag{4.7}$$

where $\Gamma_p \in \mathrm{R}^{n\times n}$ and $\gamma_p \in \mathrm{R}$ are given adaptive gains for $p \in \mathcal{M}$ and the time-varying matrix $P_p(t)$ is defined as:

$$P_p(t) = \begin{cases} P_{p,k} + \frac{P_{p,k+1}-P_{p,k}}{\delta_{k+1}}(t - t_{i,k}), & \text{for } t_{i,k} \leq t < t_{i,k+1}\\ P_{p,K}, & \text{for } t_{i,K} \leq t < t_{i+1}.\end{cases} \tag{4.8}$$

The sequence of switch-in instants of subsystem p is represented by $\{t_l^{p_\mathrm{in}}\}_{l\in\mathbb{N}^+}$, and the sequence of its switch-out instants is represented by $\{t_{l+1}^{p_\mathrm{out}}\}_{l\in\mathbb{N}^+}$. Note that the proposed adaptive law (4.7) is to be implemented as follows: at a switch-in instant of subsystem p the initial conditions of (4.7) are taken from the estimates available at the previous switch-out instant of the same subsystem, i.e., $k_p(t_l^{p_\mathrm{in}}) = k_p(t_l^{\mathrm{out}})$, and $l_p(t_l^{p_\mathrm{in}}) = l_p(t_l^{p_\mathrm{out}})$ for any $l \in \mathbb{N}^+$. Therefore, k_p and l_p evolve continuously.

Remark 4.1 Compared with adaptive laws proposed in previous research, the following considerations are in order:

- As an improvement over [91, 117], projection laws are not necessary in (4.7) due to the non-increasing behavior of the Lyapunov function at the switching instants, as will be demonstrated in the next section. Therefore, the knowledge of *a priori* bounds for $k_p(t)$ and $l_p(t)$ is not needed for (4.7).

- The adaptive laws introduced in [91, 117] derive from a classical Lyapunov function consisting of quadratic terms of the tracking error and of the parameter estimation errors, where a constant positive definite matrix P_p for each subsystem is adopted. In this chapter, we propose a new adaptive law that uses a time-varying positive definite matrix $P_p(t)$ for each subsystem.

- In [91, 117] the adaptive law is derived independently of the switching law (and vice versa). That is, the design of the switching law and of the adaptive law is decoupled. In the approach proposed here adaptive and switching laws are coupled via the solution of (4.6), which depends on the dwell time.

□

4.4 Main results

In this section, the asymptotic stability result and convergence of the parameter estimates of the proposed control scheme will be presented

4.4.1 Asymptotic stability

Theorem 4.2 *With the adaptive law* (4.7)–(4.8) *and any switching law* $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$, *the tracking error* $e(t)$ *converges to zero asymptotically as* $t \to \infty$.

Proof: Consider the following Lyapunov function:

$$V(t) = e^T(t)P_{\sigma(t)}(t)e(t) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t)\right) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\left(\tilde{l}_p^2(t)\gamma_p^{-1}\right) \tag{4.9}$$

which is continuous during any interval between two consecutive switching instants and discontinuous at switching instants considering the fact that $P_{\sigma(\cdot)}(\cdot)$ is continuous during intervals and discontinuous at switching instants. Without loss of generality, let us consider an interval $[t_i,\, t_{i+1})$ between two consecutive switching instants t_i and t_{i+1} and let $\sigma(t_i) = p$ and $\sigma(t_{i+1}) = q$ with $i \in \mathbb{N}^+$ and $p, q \in \mathcal{M}$. Then, for $t \in [t_i,\, t_{i+1})$, subsystem p is active and thus k_j and l_j for all $j \in \mathcal{M} / \{p\}$ stay constant and their values are those at the last switched-out instant of subsystem j before the time instant t_i. Therefore, using (4.8) and (4.7), the derivative of $V(t)$ with respect to time is

$$\begin{aligned} \dot{V}(t) &= \dot{e}^T(t)P_p(t)e(t) + e^T(t)P_p(t)\dot{e}(t) + e^T(t)\dot{P}_p(t)e(t) \\ &\quad + 2\frac{1}{|l_p^*|}\tilde{k}_p^T(t)\Gamma_p^{-1}\dot{\tilde{k}}_p(t) + 2\frac{1}{|l_p^*|}\tilde{l}_p(t)\dot{\tilde{l}}_p(t)\gamma_p^{-1} \\ &= e^T(t)Q_p(t)e(t) \end{aligned} \tag{4.10}$$

with $Q_p(t)$ defined as

$$Q_p(t) = A_{\mathrm{m}p}^T P_p(t) + \dot{P}_p(t) + P_p(t)A_{\mathrm{m}p} \tag{4.11}$$

which is continuous for $t \in [t_i,\, t_{i+1})$ due to the continuity of $P_p(t)$ for $t \in [t_i,\, t_{i+1})$.

To analyze the properties of $Q_p(t)$ for $t \in [t_i,\ t_{i+1})$, first we consider $t \in [t_{i,k},\ t_{i,k+1})$, $k = 0,\dots,K-1$. Note that

$$\begin{aligned} Q_p(t) &= A_{\mathrm{m}p}^T P_p(t) + (P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_p(t) A_{\mathrm{m}p} \\ &= \eta_1 \left[(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k} A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,k} \right] \\ &\quad + \eta_2 \left[(P_{p,k+1} - P_{p,k})/\delta_{k+1} + P_{p,k+1} A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_{p,k+1} \right] \end{aligned} \tag{4.12}$$

where

$$\eta_1 = 1 - \frac{t - t_{i,k}}{\delta_{k+1}}, \quad \eta_2 = \frac{t - t_{i,k}}{\delta_{k+1}}.$$

According to (4.6b) and (4.6c), it follows from (4.12) that

$$Q_p(t) < 0, \quad t \in [t_{i,k},\ t_{i,k+1}). \tag{4.13}$$

Then, let us consider $t \in [t_{i,K},\ t_{i+1})$ for the case that $t_{i+1} - t_i > \tau_{\mathrm{d}}$. We have $P_p(t) = P_{p,K}$ according to (4.8), which indicates by (4.6d) that

$$Q_p(t) = A_{\mathrm{m}p}^T P_{p,K} + P_{p,K} A_{\mathrm{m}p} < 0, \quad t \in [t_{i,K},\ t_{i+1}). \tag{4.14}$$

Therefore, it follows from (4.13)-(4.14) that $Q_p(t) < 0$ due to the continuity of $Q_p(t)$ as $t \in [t_i,\ t_{i+1})$, which implies that $V(t)$ is strictly decreasing for any $e(t) \neq 0$ for $t \in [t_i,\ t_{i+1})$, i.e.,

$$\dot{V}(t) = e^T(t) Q_p(t) e(t) < 0, \quad t \in [t_i,\ t_{i+1}). \tag{4.15}$$

Since the signals $e(\cdot)$, $\tilde{k}_{\sigma(\cdot)}(\cdot)$, and $\tilde{l}_{\sigma(\cdot)}(\cdot)$ are continuous according to (4.5) and (4.7), it follows, at switching instant t_{i+1}, that

$$\begin{aligned} &V_{\sigma(t_{i+1})}(t_{i+1}) - V_{\sigma(t_{i+1}^-)}(t_{i+1}^-) \\ &= e^T(t_{i+1}) P_{\sigma(t_{i+1})}(t_{i+1}) e(t_{l+1}) - e^T(t_{i+1}^-) P_{\sigma(t_{i+1}^-)}(t_{i+1}^-) e(t_{i+1}^-) \\ &= e^T(t_{i+1}) (P_{\sigma(t_{i+1})} - P_{\sigma(t_{i+1}^-)}) e(t_{i+1}) \\ &= e^T(t_{i+1}) \left(P_{q,0} - P_{p,K} \right) e(t_{i+1}) \end{aligned}$$

which indicates that $V(\cdot)$ is non-increasing at switching instant t_{i+1} considering $P_{q,0} - P_{p,K} \leq 0$ for $p, q \in \mathcal{M}$. Since $V(\cdot)$ is strictly decreasing during any interval between two consecutive switching instants and non-increasing at each switching instant for any $e(t) \neq 0$, now we can conclude that $V(t)$ is strictly decreasing for any $t > 0$ and $e(t) \neq 0$. This implies the boundedness of $V(\cdot)$ and therefore all the signals in the closed-loop switched system according to (4.9). Integrating (4.15) from 0 to ∞, we have

$$\begin{aligned} \int_0^\infty e^T(t) Q_p(t) e(t) dt &< V(0) - V(\infty) \\ &< \infty. \end{aligned}$$

Due to the boundedness of $P_p(\cdot)$, $Q_p(\cdot)$ is also bounded, which implies

$$\int_0^\infty e^T(t) e(t) dt < \infty$$

i.e., $e(\cdot) \in \mathscr{L}_2$. According to (4.9), since $V(\cdot) \in \mathscr{L}_\infty$, we have $e(\cdot) \in \mathscr{L}_\infty$. Additionally, the dynamics of $e(\cdot)$ in (5) gives rise to $\dot{e}(\cdot) \in \mathscr{L}_\infty$. Since $e(\cdot) \in \mathscr{L}_2$ and $\dot{e}(\cdot) \in \mathscr{L}_\infty$, it can be concluded that $e(t) \to 0$ as $t \to \infty$ according to *Barbalat's lemma* [86]. This completes the proof. □

Remark 4.3 Note that $P_p(\cdot)$ is constructed using a family of discrete matrices satisfying (4.6) for each subsystem. The computational complexity of constructing $P_p(\cdot)$ is dependent on the number K. A larger K leads to a smaller dwell time τ_d that can guarantee asymptotic tracking. However, there always exists a constant K^* such that $\forall K > K^*$, the dwell time τ_d is equivalent to the result obtained by the stability condition presented in Lemma 2.3, i.e., for $p \neq q \in \mathcal{M}$,

$$\begin{aligned} P_p, P_q &> 0 \\ P_p A_{\mathrm{m}p} + A_{\mathrm{m}p}^T P_p &< 0 \\ e^{A_{\mathrm{m}p}^T \tau_\mathrm{d}} P_q e^{A_{\mathrm{m}p}\tau_\mathrm{d}} - P_p &< 0. \end{aligned} \tag{4.16}$$

□

Remark 4.4 According to *Remark 4.3*, a question may arise automatically: why cannot we use the stability condition (4.16) directly to obtain the result of Theorem 4.2 instead of condition (4.6)? The reason is explained in the following.

The differences between the stability conditions in (4.6) and (4.16) have a significant impact on the derivative of the Lyapunov function (4.9). Note that it is not necessary to develop the derivative in (4.10) during the intervals between two consecutive switches into an exponential decay formulation, i.e., $\dot{V}(t) \leq -\alpha V(t)$ with a compatible number $\alpha > 0$, which is needed in the approach followed in [91, 117]. On the other hand, using (4.16), the following classical Lyapunov function as in [91] is considered:

$$V(t) = e^T(t) P_{\sigma(t)} e(t) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t)\right) + \sum_{p=1}^{M} \frac{1}{|l_p^*|}\left(\tilde{l}_p^2(t)\gamma_p^{-1}\right) \tag{4.17}$$

whose derivative is, for $t \in [t_i,\, t_{i+1})$

$$\begin{aligned} \dot{V}(t) &= e^T(t)(P_p A_{\mathrm{m}p} + A_{\mathrm{m}p} P_p) e(t) \\ &\quad + \frac{1}{|l_p^*|}(\tilde{k}_p^T(t)\Gamma_p^{-1} f_{xp}(t) + \gamma_i^{-1}\tilde{l}_p(t) f_{rp}(t)) \\ &\leq -\frac{V(t)}{\rho} - \frac{V(t) - \mathcal{B}}{s\rho} \end{aligned} \tag{4.18}$$

where f_{xp} and f_{rp} are projection laws, and the positive numbers ρ, s, and $\mathcal{B}$ can be calculated as shown in [91]. The derivative in (4.19) can be shown to be decreasing at an exponential rate only when $V(t) \geq \mathcal{B}$. According to condition in (4.16), asymptotic stability can be guaranteed only if the Lyapunov function in (4.17) is decreasing at an exponential rate for $t \in \mathbb{R}^+/\mathcal{S}$ which cannot be satisfied according to (4.19). In light of this, we cannot utilize (4.16) obtain the result of Theorem 4.2 due to the the presence of the exponential term $e^{A_{\mathrm{m}p}\tau_\mathrm{d}}$ in (4.16), while it does not appear in (4.6). □

Remark 4.5 In the literature on adaptive control of switched systems, the switching laws based on dwell time [91] and on average dwell time [117] are designed based on the following two properties of the Lyapunov function: an exponential decreasing rate during active intervals between two consecutive switching instants, and a bounded increment at switching instants. Note that switching laws based on these two properties and adaptive laws with constant positive definite matrices P_p prevent asymptotic tracking from being achieved, following the same reasoning as in *Remark 4.4*. □

Remark 4.6 Since subsystem matrices are necessary to calculate the dwell time using *Theorem 1* in [1], the method proposed in [1] can only guarantee asymptotic stability of switched systems with uncertainties residing in a known polytope. However, the Lyapunov function (4.9) exploits the matrices of the reference modes. As a consequence, the proposed adaptive laws (4.7) with time-varying matrices $P_p(\cdot)$ can achieve asymptotic stability of switched systems with more general (possibly non-polytopic) uncertainties. □

4.4.2 Convergence of parameter estimates

Theorem 4.7 *If the reference input signal $r(\cdot)$ is persistently exciting with respect to system* (4.2) *(i.e., $r(\cdot)$ has at least $(n+1)/2$ different frequencies), then $\tilde{k}_p(t)$, $\tilde{l}_p(t)$, $p \in \overline{\mathcal{M}}$, and $e(t)$ converge to zero asymptotically as $t \to \infty$, with the adaptive law* (4.7)–(4.8) *and any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$, where $\overline{\mathcal{M}}$ represents the set of subsystems that are active intermittently over infinite intervals.*

Proof: Define $\tilde{\theta}_p(t) = [\tilde{k}_p^T(t)\ \tilde{l}_p(t)]^T$ for $t \in T_p$, where $T_p = \cup_{l \in \mathbb{N}^+}[t_{p_l}, t_{p_l+1})$ denotes the total time period when subsystem p is active. Then, we can express (4.7) as:

$$\dot{\theta}_p(t) = -\operatorname{sgn}[l_p^*]\overline{\Gamma}_p \phi(t) b_{\mathrm{m}p}^T P_p(t) e(t)$$

where $\overline{\Gamma}_p = \operatorname{diag}\{\Gamma_p, \gamma_p\}$ and $\phi(t) = \begin{bmatrix} x^T(t) & r^T(t) \end{bmatrix}^T$. Define $\chi(t) = [e^T(t)\ \tilde{\theta}^T(t)]^T$. Then we have

$$\dot{\chi}(t) = \overline{A}_p(t)\chi(t), \quad e(t) = C_p^T \chi(t)$$

where

$$\overline{A}_p(t) = \begin{bmatrix} A_{\mathrm{m}p} & b_p \phi^T(t) \\ -\operatorname{sgn}[l_p^*]\overline{\Gamma}_p \phi(t) b_{\mathrm{m}p}^T P_p(t) & 0 \end{bmatrix}, \quad C_p = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Consider the time-varying positive definite matrix

$$\overline{P}_p(t) = \begin{bmatrix} P_p(t) & 0 \\ 0 & \frac{1}{|l_p^*|}\overline{\Gamma}_p^{-1} \end{bmatrix}$$

where $P_p(t)$ is defined in (4.8). We consider the following Lyapunov function:

$$\begin{aligned} V_p(t) &= \chi^T(t)\overline{P}_p(t)\chi(t) \\ &= e^T(t)P_p(t)e(t) + \frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t) + \tilde{l}_p^2(t)\gamma_p^{-1}\right). \end{aligned}$$

For an interval $[t_{p_l}, t_{p_l+1})$ with $l \in \mathbb{N}^+$ when subsystem p is active, the derivative of the Lyapunov function is given by

$$\begin{aligned}\dot{V}_p(t) &= \chi^T(t)\left(\overline{A}_p^T(t)\overline{P}_p(t) + \overline{P}_p(t)\overline{A}_p(t) + \dot{\overline{P}}_p(t)\right)\chi(t)\\ &= e^T(t)\left(A_{\mathrm{m}p}^T P_p(t) + \dot{P}_p(t) + P_p(t)A_{\mathrm{m}p}\right)e(t)\\ &= e^T(t)Q_p(t)e(t).\end{aligned}$$

Based on the proof of Theorem 4.2, it is clear that $V_p(t)$ is strictly decreasing for any $e(t) \neq 0$, which is equivalent to

$$\dot{V}_p(t) < 0, \quad t \in [t_{p_l}, t_{p_l+1}).$$

Therefore, there exist positive constants

$$\upsilon_p = -\sup\left\{\lambda_{\max}[Q_p(t)]\right\}$$

such that

$$\begin{aligned}\dot{V}_p(t) &\leq -\upsilon_p e^T(t)e(t)\\ &= -\upsilon_p \chi(t)^T C_p C_p^T \chi(t), \quad t \in [t_{p_l}, t_{p_l+1}).\end{aligned} \tag{4.19}$$

Furthermore, since the reference input $r(\cdot)$ is persistently exciting and $(A_{\mathrm{m}p}, b_{\mathrm{m}p})$ is controllable, it follows that $\phi_{\mathrm{m}}(\cdot) := [x_{\mathrm{m}}(\cdot)^T\ r(\cdot)^T]^T$ is also persistently exciting [7], which, together with (4.19), implies that $\phi(\cdot)$ is weakly persistently exciting (according to Definition 3 in [45]). This only leads to asymptotic convergence to zero of the system $\dot{\chi}(t) = \overline{A}_p(t)\chi(t)$ (see *Theorem 4* of [45]) for $t \in [t_{p_l}, t_{p_l+1})$. Next, we compare the value of $V_p(t)$ at the switch-out instant $t_{l+1}^{p_{\mathrm{out}}}$ and the switch-in instant $t_{l+1}^{p_{\mathrm{in}}}$ of subsystem p. Since $k_p(t_{l+1}^{p_{\mathrm{in}}}) = k_p(t_{l+1}^{p_{\mathrm{out}}})$, and $l_p(t_{l+1}^{p_{\mathrm{in}}}) = l_p(t_{l+1}^{p_{\mathrm{out}}})$, and due to the result that $V(t)$ in (4.9) is strictly decreasing for any $e(t) \neq 0$, we have

$$\begin{aligned}V_p(t_{l+1}^{p_{\mathrm{in}}}) - V_p(t_{l+1}^{p_{\mathrm{out}}}) &= V(t_{l+1}^{p_{\mathrm{in}}}) - V(t_{l+1}^{p_{\mathrm{out}}})\\ &= e(t_{l+1}^{p_{\mathrm{in}}})^T P_p(t_{l+1}^{p_{\mathrm{in}}})e(t_{l+1}^{p_{\mathrm{in}}})\\ &\quad - e(t_{l+1}^{p_{\mathrm{out}}})^T P_p(t_{l+1}^{p_{\mathrm{out}}})e(t_{l+1}^{p_{\mathrm{out}}})\\ &< 0\end{aligned} \tag{4.20}$$

which shows that $V_p(t)$ is strictly decreasing for all $t \in T_p$ together with (4.19). Now, we construct a continuous time line $\overline{t}$ by joining the intervals when subsystem p is active, i.e., taking $t_{l+1}^{p_{\mathrm{in}}} = t_{l+1}^{p_{\mathrm{out}}}$ for all $l \in \mathbb{N}^+$ and $\overline{t}_0 = t_{p_0}$. Therefore, according to (4.19)–(4.20), it holds that the system $\dot{\chi}(\overline{t}) = \overline{A}_p(t)\chi(\overline{t})$ is asymptotically stable for the time line $\overline{t}$, that is, $\chi(\overline{t}) \to 0$ as $\overline{t} \to \infty$, which indicates that $e(t)$, $\tilde{k}_p(t)$, and $\tilde{l}_p(t)$ converge to zero asymptotically, as $t \in T_p \to \infty$. This completes the proof. □

4.5 Example

In this section, an electro-hydraulic system [97, 108], as shown in Fig. 4.2, is used to demonstrate the effectiveness of the proposed adaptive asymptotic tracking control scheme. The sensors Linear Variable Differential Transformer (LVDT) and Velocity Receiver are measuring the displacement and velocity.

Two operating conditions with respect to different supply pressures, 11.0 MPa and 1.4

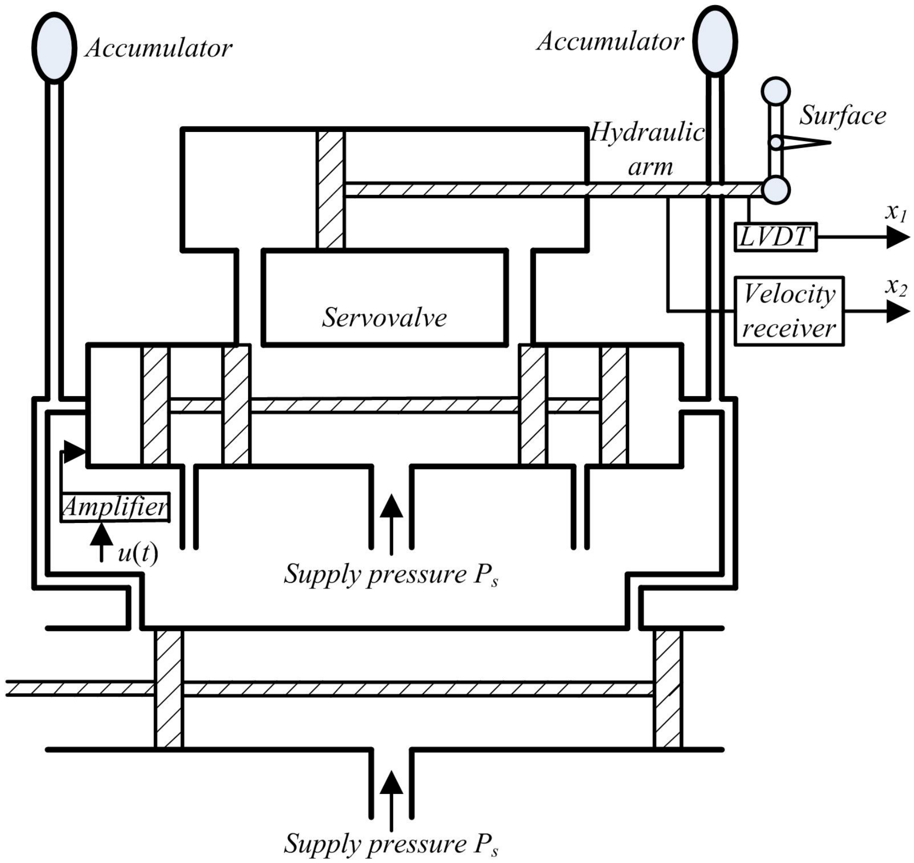

Figure 4.2: The schematic diagram of the electro-hydraulic system.

MPa, are selected, and the corresponding transfer functions are:

$$G_1(s) = \frac{62.4}{s(s+4.58)}, \quad G_2(s) = \frac{47.2}{s(s+9.19)}$$

which can be written in canonical form:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -4.58 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 62.4 \end{bmatrix} u(t), \quad 11.0 \text{ MPa}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -9.19 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 47.2 \end{bmatrix} u(t), \quad 1.4 \text{ MPa}$$

where $x = [x_1 \ x_2]^T$ with x_1, x_2 representing the displacement of the arm and the velocity of the arm, respectively. The input $u(t)$ is the control voltage.

The desired behavior is represented by:

$$\dot{x}_{\mathrm{m}}(t) = \begin{bmatrix} 0 & 1 \\ -15 & -8 \end{bmatrix} x_{\mathrm{m}}(t) + \begin{bmatrix} 0 \\ 31.2 \end{bmatrix} r(t), \quad 11.0 \text{ MPa}$$

$$\dot{x}_{\mathrm{m}}(t) = \begin{bmatrix} 0 & 1 \\ -27 & -12 \end{bmatrix} x_{\mathrm{m}}(t) + \begin{bmatrix} 0 \\ 23.6 \end{bmatrix} r(t), \quad 1.4 \text{ MPa}.$$

With $K = 1$ we have a dwell time $\tau_{\mathrm{d}} = 1$ and the matrices obtained by solving the LMIs in (4.6) are:

$$P_{1,0} = \begin{bmatrix} 1.2605 & 0.0546 \\ 0.0546 & 0.0540 \end{bmatrix}, \quad P_{1,1} = \begin{bmatrix} 2.0107 & 0.0491 \\ 0.0491 & 0.1216 \end{bmatrix}$$

$$P_{2,0} = \begin{bmatrix} 1.3832 & 0.0305 \\ 0.0305 & 0.0443 \end{bmatrix}, \quad P_{2,1} = \begin{bmatrix} 2.1008 & 0.0279 \\ 0.0279 & 0.0818 \end{bmatrix}.$$

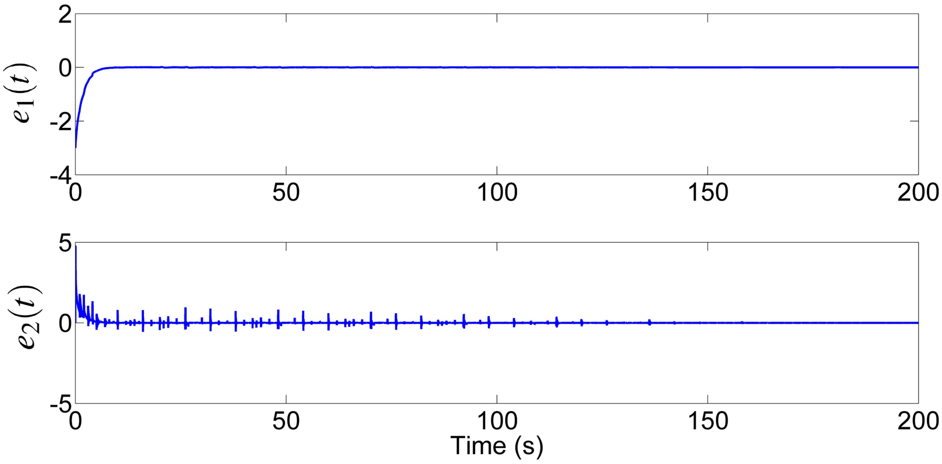

Figure 4.3: The tracking error.

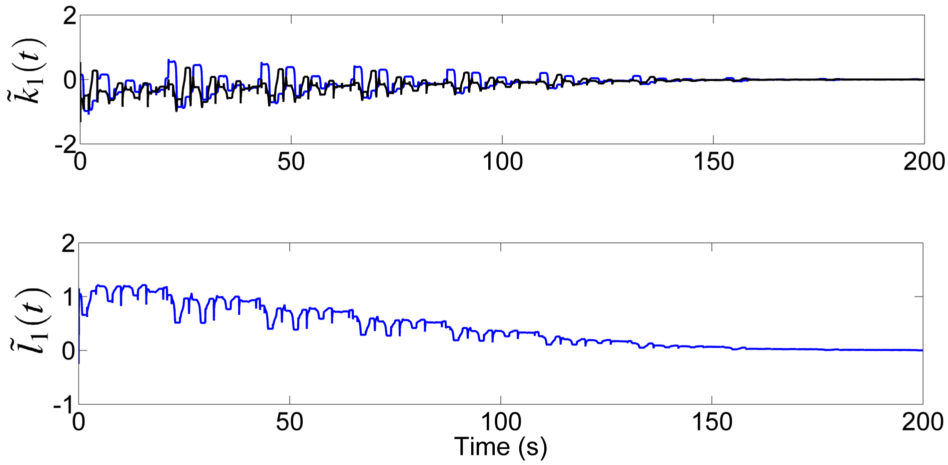

Figure 4.4: The parameter estimation errors of the controller for subsystem 1.

Without loss of generality, we select the switching interval $t_{i+1} - t_i = \tau_{\mathrm{d}}$, for all i, of the switching law $\sigma(\cdot)$. Therefore, the time-varying positive matrix $P_p(t)$ for $p \in \{1,2\}$ can be calculated by $P_p(t) = (t - \tau_{\mathrm{d}} \cdot \mathrm{floor}(t/\tau_{\mathrm{d}})) \cdot \left(P_{p,2} - P_{p,1}\right)/\tau_{\mathrm{d}} + P_{p,1}$, where floor$(t/\tau_{\mathrm{d}})$ rounds t/τ_{d} to the nearest integer less than or equal to t/τ_{d}. The initial conditions are chosen as: $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $x_{\mathrm{m}}(0) = \begin{bmatrix} 3 & 0 \end{bmatrix}^T$, $l_p(0) = 0.5 l_p^*$ and $k_p(0) = 0.5 k_p^*$, $p \in \{1,2\}$. The adaptive gains $\Gamma_p = 10I$, $\gamma_p = 5$, $p \in \{1,2\}$, are selected. In addition, a persistently exciting reference input $r(t) = 3\sin(\pi t) + 2\cos(2t)$ is chosen. The simulation results are shown in Figs. 4.3–(4.5), which indicate that the tracking error converges to 0 and the parameter estimates of

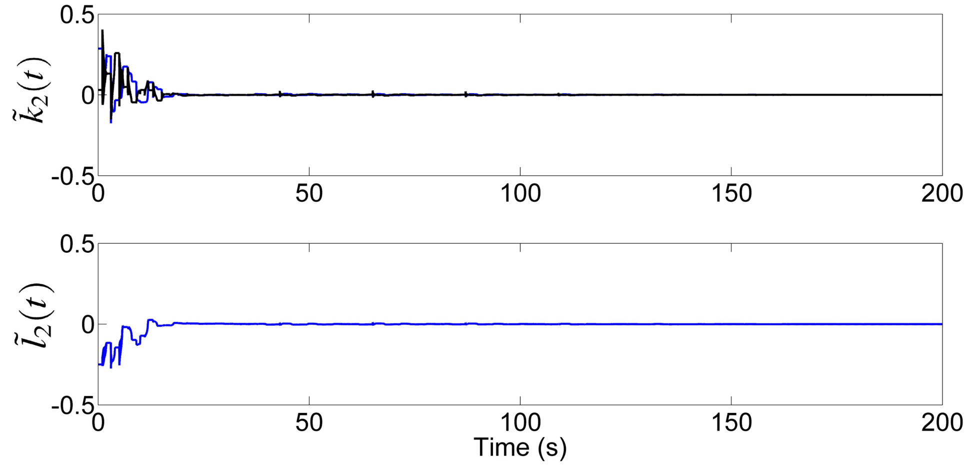

Figure 4.5: The parameter estimation errors of the controller for subsystem 2.

the controller $k_1(t)$, $l_1(t)$, $k_2(t)$ and $l_2(t)$ converge to $k_1^* = \begin{bmatrix} -0.2404 & -0.0548 \end{bmatrix}^T$, $l_1^* = 0.5$, $k_2^* = \begin{bmatrix} -0.5720 & -0.0595 \end{bmatrix}^T$, and $l_2^* = 0.5$ asymptotically, respectively.

4.6 Concluding remarks

In this chapter, the adaptive asymptotic tracking problem of uncertain switched linear systems has been investigated. An adaptive law based on a time-varying positive definite matrix and a dwell time switching law have been developed. The proposed control scheme can guarantee the asymptotic stability of the tracking error. Furthermore, with the presence of a persistently exciting reference input, the parameter estimates of the controller converge to the real values asymptotically. In light of this, the proposed method has filled the theoretical gap between adaptive control of switched linear systems and non-switched linear systems. A practical example of an electro-hydraulic system has demonstrated the effectiveness of the proposed adaptive control scheme.

Chapter 5

Robust Adaptive Tracking of Switched Linear Systems with Dwell Time

This chapter investigates robust adaptive control of uncertain switched linear systems considering disturbances. Two robustification designs for the adaptive law in Chapter 4 are proposed based on parameter projection and on leakage. The main difference of the designs consists in the available *a priori* knowledge: the projection law requires knowledge of the bounds of the parameter estimates while the leakage law does not require knowledge of the bounds of the parameter estimates. The closed-loop switched linear system is shown to be globally uniformly ultimately bounded, i.e., robust adaptation is achieved in the presence of disturbances. In addition, the ultimate bounds of both adaptive control schemes are given.

Parts of the research presented in the chapter have been published in [134].

5.1 Introduction

As parametric uncertainties have been addressed in the previous chapters, another ubiquitous problem for switched linear systems is how to achieve robustness when considering external disturbances. It is well established that robust control can be used to deal with non-switched systems subject to uncertainties and disturbances [109, 110]. To date, robust control of switched systems has been extensively investigated: a single robust controller has been adopted [65, 150], and a family of robust controllers for polytopic uncertainties have been designed [1, 2]. However, a single robust controller may lead to conservative performance when considering large uncertainties [95]. As a complement to robust control, adaptive control techniques have been shown to be able to deal with large non-polytopic uncertainties and disturbances [4, 44]. However, to the best of the author's knowledge, the research on robust adaptive control of switched systems considering both parametric uncertainties and disturbances is a quite open field. In [108], Qing *et al.* proposed a robust adaptive control scheme for switched linear systems that requires the existence of a common Lyapunov function. Hidetoshi and Kojiro developed a so-called adaptive robust controller for a family of switched linear systems subject to parametric uncertainties, which did not address disturbances [83]. In light of this, this chapter aims to develop a robust adaptive control scheme to deal with switched linear systems considering non-polytopic parametric uncertainties and disturbances.

In this chapter, the results about adaptive control of switched linear systems without considering disturbances in Chapter 4 are exploited to develop two robust adaptive control schemes for switched linear systems. With an assumption on the knowledge of the bounds of nominal parameters, a robust adaptive control scheme using parameter projection is proposed, which is an immediate extension of the result in Section 8.5.5 of [43]. Next, a robust adaptive control scheme is developed via a leakage approach involving initial conditions of the parameter estimates; this approach is different from the results established in Section 8.5.2 of [43]: the leakage terms involve the difference between the parameter estimates and the initial conditions. In addition, the closed-loop switched linear system is shown to be globally uniformly ultimately bounded based on the proposed two robust adaptive schemes, and ultimate bounds for both cases are also given.

This chapter is organized as follows: Section 5.2 presents the control problem and some preliminaries for later analysis. In Section 5.3, we introduce robust adaptive control schemes based on a projection law and a leakage approach, respectively. In addition, the results about global uniform ultimate boundedness of the closed-loop switched linear systems are also given in 5.4. Section 5.5 adopts two examples to illustrate the proposed results. The chapter is concluded in Section 5.6.

5.2 Problem statement

This chapter focuses on the uncertain single-input switched linear system with a bounded disturbance defined as follows:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b_{\sigma(t)}u(t) + d(t), \quad t \geq t_0 \tag{5.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ represents some piecewise continuous input, and $d(\cdot)$ is a bounded disturbance with an upper bound $\overline{d}$. The switching signal $\sigma : [0,\infty) \to \mathcal{M} := \{1,2,\ldots,M\}$ is a piecewise function with M denoting the number of subsystems. The matrices $A_p \in \mathbb{R}^{n\times n}$ and vectors $b_p \in \mathbb{R}^n$ are unknown for all $p \in \mathcal{M}$.

A reference switched system representing the desired behavior of (5.1) is given as follows:

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}\sigma(t)}x_{\mathrm{m}}(t) + b_{\mathrm{m}\sigma(t)}r(t), \quad t \geq t_0 \tag{5.2}$$

where $x_{\mathrm{m}} \in \mathbb{R}^n$ is the desired state vector, and $r \in \mathbb{R}$ is a bounded reference input. The matrices $A_{\mathrm{m}p} \in \mathbb{R}^{n\times n}$ and vectors $b_{\mathrm{m}p} \in \mathbb{R}^n$ are known, and $A_{\mathrm{m}p}$ are Hurwitz matrices for all $p \in \mathcal{M}$. Suppose that the pair $(A_{\mathrm{m}p},\ b_{\mathrm{m}p})$ is controllable and each subsystem in (5.1) has its own corresponding reference submodel. We assume that the measurement of the state $x(t)$ is available. Hence, the nominal state feedback controller that makes the switched system behave like the reference model is given as $u^*(t) = k^{*T}_{\sigma(t)}x(t) + l^*_{\sigma(t)}r(t)$ for $t \geq t_0$, where the nominal parameters $k^*_p \in \mathbb{R}^n$ and $l^*_p \in \mathbb{R}$ exist under the assumption that the following matching condition holds [43, 91]:

$$A_p + b_p k^{*T}_p = A_{\mathrm{m}p}, \quad b_p l^*_p = b_{\mathrm{m}p}. \tag{5.3}$$

However, since A_p and b_p are unknown, we cannot calculate k^*_p and l^*_p from (5.3). In light of this, the state feedback controller is developed as:

$$u(t) = k^T_{\sigma(t)}(t)x(t) + l_{\sigma(t)}(t)r(t), \quad t \geq t_0 \tag{5.4}$$

where k_p and l_p are the estimates of k_p^* and l_p^* respectively. In addition, we define the tracking error as $e(t) = x(t) - x_{\mathrm{m}}(t)$. Substituting (5.4) into (5.1), and subtracting (5.2), the dynamics of the tracking error are given as follows:

$$\dot{e}(t) = A_{\mathrm{m}\sigma(t)}e(t) + b_{\sigma(t)}\left(\tilde{k}_{\sigma(t)}^T(t)x(t) + \tilde{l}_{\sigma(t)}(t)r(t)\right) + d(t) \tag{5.5}$$

where $\tilde{k}_p = k_p - k_p^*$, $\tilde{l}_p = l_p - l_p^*$ are the parameter estimation errors.

Thus, the problem addressed in this chapter is presented as follows:

Problem 5.1 *Develop a switching law $\sigma(\cdot)$ based on dwell time and an adaptive law for k_p and l_p such that the switched system* (5.1) *with the state-feedback controller* (5.4) *is stable, and the tracking error is globally uniformly ultimately bounded.*

5.3 Design of robust adaptive controllers

The following lemma is given to derive the main results in this section.

Lemma 5.1 *[129] Let $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, and Φ, Ψ be appropriately dimensioned matrices, then for any positive constant ϵ and for any appropriately dimensioned matrix $X(t)$ satisfying $X^T(t)X(t) \leq I$, it holds that*

$$2y^T\Phi X(t)\Psi z \leq \epsilon y^T\Phi\Phi^T y + \epsilon^{-1}z^T\Psi^T\Psi z.$$

5.3.1 Switching laws

Let K be a given integer. Let us define a time sequence $\{t_{i,k}\}_{k=0}^K$ with $t_{i,k+1} - t_{i,k} = h$, $k = 0,\ldots, K-1$. We define that $t_{i,0} = t_i$ and $t_{i,K} - t_{i,0} = \tau_{\mathrm{d}}$, as shown in Fig. 7.3. Suppose that there

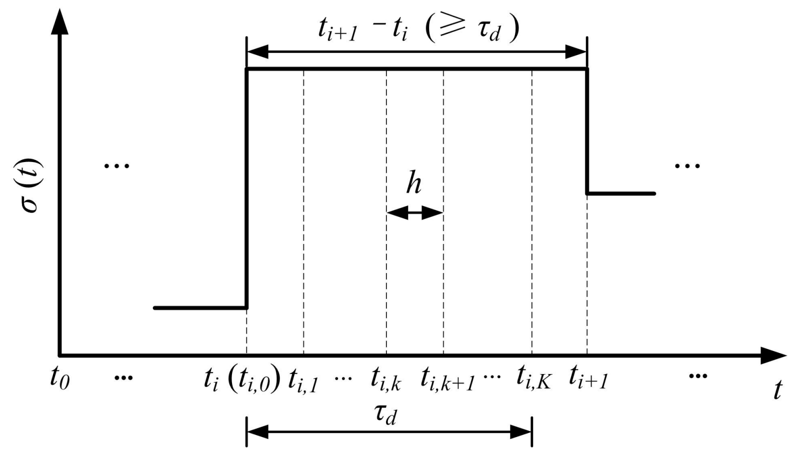

Figure 5.1: The time sequence between two consecutive switching instants.

exists a family of matrices $P_{p,k} > 0$, $p \in \mathcal{M}$, $k = 0,\ldots,K$and a family of positive constants κ_p,

$p \in \mathcal{M}$ such that the following conditions hold:

$$A_{\mathrm{m}p}^T P_{p,k} + P_{p,k} A_{\mathrm{m}p} + \frac{P_{p,k+1} - P_{p,k}}{h} + \left(1+\kappa_p\right) P_{p,k} < 0 \tag{5.6a}$$

$$A_{\mathrm{m}p}^T P_{p,k+1} + P_{p,k+1} A_{\mathrm{m}p} + \frac{P_{p,k+1} - P_{p,k}}{h} + \left(1+\kappa_p\right) P_{p,k+1} < 0 \tag{5.6b}$$

$$\text{for } k = 0, \ldots, K-1$$

$$A_{\mathrm{m}p}^T P_{p,K} + P_{p,K} A_{\mathrm{m}p} + \left(1+\kappa_p\right) P_{p,K} < 0 \tag{5.6c}$$

$$P_{p,K} - P_{q,0} \geq 0. \tag{5.6d}$$

Then, the adaptive laws based on parameter projections and leakage approach will be developed based on the family of the matrices $P_{p,k} > 0$, $p \in \mathcal{M}$, $k = 0, \ldots, K$, and a switching law is proposed based on the following dwell time:

$$\tau_{\mathrm{d}} = Kh. \tag{5.7}$$

Remark 5.1 The selection of K is dependent on rule proposed in [122]: as K grows, smaller (less conservative) h values can be found by solving the LMIs (5.6). In addition, there exists an integer K^* such that no less conservative h can be obtained by choosing a sufficiently large $K \geq K^*$. □

5.3.2 Adaptive laws using parameter projection

Before introducing the adaptive law, the following assumptions are made:

Assumption 5.1 *The sign of* l_p^*, $\forall p \in \mathcal{M}$, *is known.*

Assumption 5.2 *Upper and lower bounds of* k_p^* *and* l_p^* *are known, i.e.,* $k_p^* \in [\underline{k}_p,\ \overline{k}_p]$ *and* $l_p^* \in [\underline{l}_p,\ \overline{l}_p]$, $\forall p \in \mathcal{M}$.

Remark 5.2 Assumptions 5.1 and 5.2 are widely used in adaptive control based on parameter projections [91, 118] to ensure the boundedness of signals in closed-loop systems [43]. □

The adaptive law with the following projection laws is proposed:

$$\begin{aligned} \dot{k}_p(t) &= -\operatorname{sgn}[l_p^*]\Gamma_p x(t) e^T(t) P_p(t) b_{\mathrm{m}p} + f_{k,p}(t) \\ \dot{l}_p(t) &= -\operatorname{sgn}[l_p^*]\gamma_p r(t) e^T(t) P_p(t) b_{\mathrm{m}p} + f_{l,p}(t) \end{aligned} \tag{5.8}$$

where $\Gamma_p \in \mathbb{R}^{n\times n}$ and $\gamma_p \in \mathbb{R}$ are given positive adaptive gains for all $p \in \mathcal{M}$ and the time-varying matrix $P_p(t)$ is defined as:

$$P_p(t) = \begin{cases} P_{p,k} + \frac{t - t_{i,k+1}}{h}\left(P_{p,k+1} - P_{p,k}\right), & \text{for } t_{i,k} \leq t < t_{i,k+1} \\ P_{p,K}, & \text{for } t_{i,K} \leq t < t_{i+1}. \end{cases} \tag{5.9}$$

The functions $f_{k,p}(\cdot)$ and $f_{l,p}(\cdot)$ are the projection laws, which are used to prevent *parameter drift* of the parameter estimates [44]. Next, the definitions of $f_{k,p}$ and $f_{l,p}$ are given [118]: Let

$k_p = [k_{p1}, \ldots, k_{pn}]$; $f_{k,p} = [f_{k1,p}, \ldots, f_{kn,p}]$;

$$\begin{aligned}\phi_{k,p} &= -\operatorname{sgn}[l_p^*]\Gamma_p x e^T P_p b_{\mathrm{m}} = [\phi_{k1,p}, \ldots, \phi_{kn,p}] \\ \phi_{l,p} &= -\operatorname{sgn}[l_p^*]\gamma_p r e^T P_p b_{\mathrm{m}p}.\end{aligned}$$

Then, we have the projection terms as follows, for $s \in \{1, \cdots, n\}$, $t \geq t_0$

$$\begin{aligned} f_{ks,p}(t) &= \begin{cases} -\phi_{ks,p}(t) & \text{if } k_{ps}(t) \leq \underline{k}_{ps} \;\&\; \phi_{ks,p}(t) \leq 0, \\ & \text{or if } k_{ps}(t) \geq \overline{k}_{ps} \;\&\; \phi_{ks,p}(t) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\ f_{l,p}(t) &= \begin{cases} -\phi_{l,p}(t) & \text{if } l_p(t) \leq \underline{l}_p \;\&\; \phi_{l,p}(t) \leq 0, \\ & \text{or if } l_p(t) \geq \overline{k}_p \;\&\; \phi_{l,p}(t) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.10}$$

5.3.3 Adaptive laws using leakage method

Now, the leakage approach in [43] is extended to switched linear systems to prevent parameter drift, which does not require *Assumption* (5.2). The resulting adaptive law is given in the following:

$$\dot{k}_p(t) = -\operatorname{sgn}[l_p^*]\Gamma_p x(t) e^T(t) P_p(t) b_{\mathrm{m}p} - \delta_p^k \Gamma_p \hat{k}_p(t) \tag{5.11a}$$

$$\dot{l}_p(t) = -\operatorname{sgn}[l_p^*]\gamma_p r(t) e^T(t) P_p(t) b_{\mathrm{m}p} - \delta_p^l \gamma_p \hat{l}_p(t) \tag{5.11b}$$

$$\dot{k}_q(t) = -\delta_q^k \Gamma_q \hat{k}_p(t) \tag{5.11c}$$

$$\dot{l}_q(t) = -\delta_q^l \gamma_q \hat{l}_p(t) \tag{5.11d}$$

for $q = 1, \ldots, p-1, p+1, \ldots, M$, where $\hat{k}_p(t) = k_p(t) - k_p(t_0)$, $\hat{l}_p(t) = l_p(t) - l_p(t_0)$, $P_p(t)$ is defined in (5.9), Γ_p, γ_p are positive adaptive gains, and δ_p^k, δ_p^l are positive leakage rates satisfying

$$\delta_p^k \geq \max_{p\in\mathcal{M}} \{\kappa_p\}\, \lambda_{\max}\left(\Gamma_p^{-1}\right), \quad \delta_p^l \geq \max_{p\in\mathcal{M}} \{\kappa_p\}\, \gamma_p^{-1}. \tag{5.12}$$

Remark 5.3 Different from the adaptive law (5.11a), the parameter estimates of all subsystems are updated during the whole time horizon. To be more precise, the updating rules (5.11a)—(5.11b) of the parameter estimates are adopted when the subsystem is active. The updating rules switch to (5.11c)—(5.11d) when the subsystem is inactive. In addition, different from the leakage approach in [43] for adaptive control of non-switched systems, when a subsystem is inactive, the adaptive rule will bring the parameter estimates to initial conditions of (5.11) to guarantee the stability of the switched systems. □

5.4 Main results

5.4.1 Performance analysis with parameter projection

Now, we are ready to introduce the following stability results.

Theorem 5.4 *With any switching law* $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ *and the adaptive law* (5.8)–(5.10)*, the uncertain switched system* (5.1) *with state feedback controller* (5.4) *is GUUB. In addition, the ultimate bound of the tracking error is given as*

$$\mathscr{B}_{\mathrm{proj}} = \sqrt{\frac{\max_{p\in\mathcal{M}}\left\{\lambda_{\max}\left(P_p(t)\right)\right\}}{\min_{p\in\mathcal{M}}\left\{\kappa_p\right\}\min_{p\in\mathcal{M}}\left\{\lambda_{\min}\left(P_p(t)\right)\right\}}}\|\overline{d}\|. \tag{5.13}$$

Proof: Consider the following Lyapunov function:

$$V(t) = e^T(t)P_{\sigma(t)}(t)e(t) + \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t)\right) + \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\tilde{l}_p^2(t)\gamma_p^{-1}\right). \tag{5.14}$$

Without loss of generality, we assume that subsystem p is active for $t \in [t_i, t_{i+})$, $i \in \mathbb{N}^+$. Therefore, using (5.8) and (5.9), the derivative of $V(t)$ with respect to time is, for $t \in [t_i, t_{i+1})$

$$\begin{aligned}\dot{V}(t) &= e^T(t)Q_p(t)e(t) + d^T(t)P_p(t)e(t) + e^T P_p(t)d(t)\\ &\quad + \frac{1}{|l_p^*|}\tilde{k}_p^T\Gamma_p^{-1}f_{k,p}(t) + \frac{1}{|l_p^*|}\tilde{l}_p^T\gamma_p^{-1}f_{l,p}(t)\end{aligned} \tag{5.15}$$

with $Q_p(t) = A_{\mathrm{m}p}^T P_p(t) + \dot{P}_p(t) + P_p(t)A_{\mathrm{m}p}$. According to (5.10), we have $\tilde{k}_p^T\Gamma_p^{-1}f_{k,p} \le 0$, and $\tilde{l}_p\gamma_{1p}^{-1}f_{l,p} \le 0$. Since $P_p(\cdot)$ is a positive definite matrix, there exists a non-singular matrix $H_p(\cdot)$ such that $P_p(\cdot) = H_p(\cdot)H_p^T(\cdot)$. Then, substituting $P_p(\cdot) = H_p(\cdot)H_p^T(\cdot)$ into (5.15), according to *Lemma 5.1*, it follows that

$$\begin{aligned}\dot{V}(t) &\le e^T(t)Q_p(t)e(t) + e^T(t)P_p(t)e(t) + d^T(t)P_p(t)d(t)\\ &\le e^T(t)\Xi_p(t)e(t) + d^T(t)P_p(t)d(t)\end{aligned} \tag{5.16}$$

where $\Xi_p(t) = Q_p(t) + P_p(t)$. To analyze the properties of $V(t)$ for $t \in [t_i, t_{i+1})$, first we consider a subinterval, i.e., $t \in [t_{i,k}, t_{i,k+1})$, $k = 0, \ldots, K-1$. Note that

$$\begin{aligned}\Xi_p(t) &= A_{\mathrm{m}}^T P_p(t) + P_p(t)A_{\mathrm{m}p} + \frac{P_{p,k+1} - P_{p,k}}{h}\\ &\quad + \left(\eta_1(t)P_{p,k} + \eta_2(t)P_{p,k+1}\right)\\ &= \eta_1(t)\left(\frac{P_{p,k+1} - P_{p,k}}{h} + A_{\mathrm{m}p}^T P_{p,k} + P_{p,k}A_{\mathrm{m}p} + P_{p,k}\right)\\ &\quad + \eta_2(t)\left(\frac{P_{p,k+1} - P_{p,k}}{h} + A_{\mathrm{m}p}^T P_{p,k+1} + P_{p,k+1}A_{\mathrm{m}p} + P_{p,k+1}\right)\end{aligned} \tag{5.17}$$

where $\eta_1(t) = 1 - \left(t - t_{i,k}\right)/h$, and $\eta_2(t) = 1 - \eta_1(t)$. According to (5.6a)–(5.6c), it follows that

$$\Xi_p(t) + \kappa_p P_p(t) < 0, \quad t \in [t_{i,k}, t_{i,k+1}). \tag{5.18}$$

Then, let us consider $t \in [t_{i,K}, t_{i+1})$ for the case that $t_{i+1} - t_i > \tau_{\mathrm{d}}$. We have $P_p(t) = P_{p,K}$ according to (5.9), which indicates by (5.6d) that

$$\Xi_p(t) + \kappa_p P_{p,K} < 0, \quad t \in [t_{i,K}, t_{i+1}). \tag{5.19}$$

Therefore, it follows from (5.18)–(5.19) that $\Xi_p(t) < -\kappa_p P_p(t)$ for $t \in [t_i,\, t_{i+1})$. In light of this, according to (5.16), we have

$$\dot{V}(t) \le -\kappa_p e^T P_p(t) e(t) + d^T(t) P_p(t) d(t),\ t \in [t_i,\, t_{i+1}). \tag{5.20}$$

Since the signals $e(\cdot)$, $\tilde{k}_{\sigma(\cdot)}(\cdot)$, and $\tilde{l}_{\sigma(\cdot)}(\cdot)$ are continuous according to (5.5) and (5.8), we have, at switching instant t_{i+1},

$$\begin{aligned} & V_{\sigma(t_{i+1})}(t_{i+1}) - V_{\sigma(t_{i+1}^-)}(t_{i+1}^-) \\ &= e^T(t_{i+1}) P_{\sigma(t_{i+1})}(t_{i+1}) e(t_{i+1}) - e^T(t_{i+1}^-) P_{\sigma(t_{i+1}^-)}(t_{i+1}^-) e(t_{i+1}^-) \\ &= e^T(t_{i+1})(P_{\sigma(t_{i+1})} - P_{\sigma(t_{i+1}^-)}) e(t_{i+1}) \\ &= e^T(t_{i+1}) \left(P_{q,0} - P_{p,K}\right) e(t_{i+1}) \end{aligned} \tag{5.21}$$

which indicates that $V(\cdot)$ is non-increasing at switching instant t_{i+1} considering $P_{p,0} - P_{q,K} \le 0$ for $p, q \in \mathcal{M}$. Therefore, according to (5.20)-(5.21), it can be shown that there exists a ball centered at the origin with the following radius:

$$\mathscr{B}_{\text{proj}} = \sqrt{\frac{\max_{p\in\mathcal{M}} \{\lambda_{\max}\left(P_p(t)\right)\}}{\min p \in \mathcal{M} \{\kappa_p\} \min_{p\in\mathcal{M}} \{\lambda_{\min}\left(P_p(t)\right)\}}}\, \|\bar{d}\|$$

such that when $\|e(t)\| \ge \mathscr{B}_{\text{proj}}$, we have $\dot{V}(t) < 0$. Furthermore, since the parameter estimates are bounded due to the projection laws (5.10), $V(\cdot)$ is GUUB, and the tracking error $e(\cdot)$ is attracted into the ball centered in the origin with radius $\mathscr{B}_{\text{proj}}$, as shown in Fig. 5.2. This completes the proof.

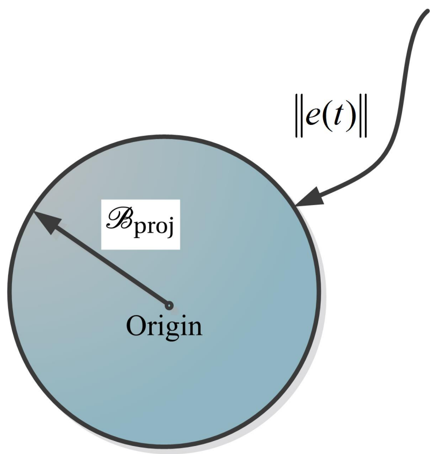

Figure 5.2: The attraction of the tracking error based on parameter projection.

□

5.4.2 Performance analysis with leakage

The following stability result is given based on the adaptive law (5.11)–(5.12), and the switching signals with dwell time τ_{d} defined in (5.7).

Theorem 5.5 *With any switching law* $\sigma(\cdot) \in \mathcal{D}(\tau_{\text{d}})$ *and the adaptive law* (5.11)–(5.12), *the* *uncertain switched system* (5.1) *with state-feedback controller* (5.4) *is GUUB. In addition, the ultimate bound of the tracking error is given as*

$$\mathscr{B}_{\text{leak}} = \sqrt{\frac{\Sigma + \max_{p\in\mathcal{M}}\left\{\lambda_{\max}\left(P_p(t)\right)\right\}\|\overline{d}\|^2}{\min_{p\in\mathcal{M}}\left\{\kappa_p\right\}\min_{p\in\mathcal{M}}\left\{\lambda_{\min}\left(P_p(t)\right)\right\}}} \tag{5.22}$$

with

$$\Sigma = \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\delta_p^k\|k_p^* - k_p(t_0)\|^2 + \delta_p^l\left(l_p^* - l_p(t_0)\right)^2\right). \tag{5.23}$$

Proof: Consider the same Lyapunov function as in (5.14). Using (5.5), (5.6), and (5.11), the derivative of $V(t)$ w.r.t. time is, for $t \in [t_i, t_{i+1})$,

$$\begin{aligned}
\dot{V}(t) &= e^T(t)Q_{\sigma(t_i)}(t)e(t) + d^T(t)P_{\sigma(t_i)}(t)e(t) + e^T(t)P_{\sigma(t_i)}(t)d(t) - 2\sum_{p=1}^{M}\frac{1}{|l_p^*|}\delta_p^k\tilde{k}_p^T(t)\hat{k}_p(t)\\
&\quad - 2\sum_{p=1}^{M}\frac{1}{|l_p^*|}\delta_p^l\tilde{l}_p(t)\hat{l}_p(t)\\
&\le e^T(t)\Xi_{\sigma(t_i)}(t)e(t) + d^T(t)P_{\sigma(t_i)}(t)d(t) - 2\sum_{p=1}^{M}\frac{1}{|l_p^*|}\delta_p^k\tilde{k}_p^T(t)(\tilde{k}_p(t) + k_p^* - k_p(t_0))\\
&\quad - 2\sum_{p=1}^{M}\frac{1}{|l_p^*|}\delta_p^l\tilde{l}_p(t)(\tilde{l}_p(t) + l_p^* - l_p(t_0))\\
&\le -\kappa_{\sigma(t_i)}e^T(t)P_{\sigma(t_i)}(t)e(t) + d^T(t)P_{\sigma(t_i)}(t)d(t) - \sum_{p=1}^{M}\frac{1}{|l_p^*|}\delta_p^k\left(\|\tilde{k}_p(t)\|^2 - \|k_p^* - k_p(t_0)\|^2\right)\\
&\quad - \sum_{p=1}^{M}\frac{1}{|l_p^*|}\delta_p^l\left(\tilde{l}_p^2(t) - \left(l_p^* - l_p(t_0)\right)^2\right).
\end{aligned} \tag{5.24}$$

The first inequality in (5.24) holds by following the same steps as for (5.17)–(5.19), and the second inequality holds since $-2\tilde{k}_p^T\tilde{k}_p - 2\tilde{k}_p^T\left(k_p^* - k_p(t_0)\right) \le -\|\tilde{k}_p\|^2 + \|k_p^* - k_p(t_0)\|^2$, and $-2\tilde{l}^2 - 2\tilde{l}_p\left(l_p^* - l_p(t_0)\right) \le -\tilde{l}_p^2 + \left(l_p^* - l_p(t_0)\right)^2$, $\forall p \in \mathcal{M}$. Hence, according to (5.14), the following holds:

$$\begin{aligned}
\dot{V}(t) &\le -\kappa_{\sigma(t_i)}V(t) + d^T(t)P_{\sigma(t_i)}(t)d(t) + \kappa_{\sigma(t_i)}\sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\tilde{k}_p^T(t)\Gamma_p^{-1}\tilde{k}_p(t) + \tilde{l}_p^2(t)\gamma_p^{-1}\right)\\
&\quad - \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\delta_p^k\|\tilde{k}_p(t)\|^2 - \delta_p^k\|k_p^* - k_p(t_0)\|^2\right)\\
&\quad - \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left(\delta_p^l\tilde{l}_p^2(t) - \delta_p^l\left(l_p^* - l_p(t_0)\right)^2\right)\\
&\le -\kappa_{\sigma(t_i)}V(t) + d^T(t)P_{\sigma(t_i)}(t)d(t) + \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left[\kappa_{\sigma(t_i)}\lambda_{\max}\left(\Gamma_p^{-1}\right) - \delta_p^k\right]\|\tilde{k}_p(t)\|^2\\
&\quad + \sum_{p=1}^{M}\frac{1}{|l_p^*|}\left[\kappa_{\sigma(t_i)}\gamma_p^{-1} - \delta_p^l\right]\tilde{l}_p^2(t) + \Sigma
\end{aligned} \tag{5.25}$$

where Σ is defined as (5.23). According to (5.12), (5.25) is recast into

$$\dot{V}(t) \leq -\kappa_{\sigma(t_i)} V(t) + d^T(t) P_{\sigma(t_i)}(t) d(t) + \Sigma. \tag{5.26}$$

Due to the same reason as for (5.21), $V(\cdot)$ is non-increasing at the switching instants. In light of this, the Lyapunov function is attracted inside a ball centered at the origin with radius

$$\mathscr{B}_{\mathrm{V}} = \frac{1}{\min_{p\in\mathcal{M}}\{\kappa_p\}} \left(\Sigma + \max_{p\in\mathcal{M}} \{\lambda_{\max}(P_p(t))\} \|\overline{d}\|^2 \right).$$

This implies that the switched system (5.1) with state-feedback controller (5.4) is GUUB. Considering that $\|e(t)\|^2 \leq V(t)/\min_{p\in\mathcal{M}}\{\lambda_{\min}(P_p(t))\}$, it can be shown that the tracking error is attracted inside a ball centered at the origin with the following radius,

$$\mathscr{B}_{\mathrm{leak}} = \sqrt{\frac{\Sigma + \max_{p\in\mathcal{M}}\{\lambda_{\max}(P_p(t))\} \|\overline{d}\|^2}{\min_{p\in\mathcal{M}}\{\kappa_p\} \min_{p\in\mathcal{M}}\{\lambda_{\min}(P_p(t))\}}}$$

as shown in Fig. 5.3. This completes the proof.

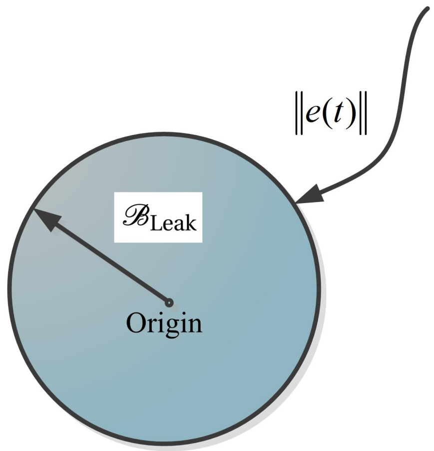

Figure 5.3: The attraction of the tracking error based on the leakage method.

□

Remark 5.6 Note that the ultimate bound of the tracking error depends on the initial parameter estimate errors: when the initial estimates are far away from the nominal parameters, a large tracking error is expected, and vice versa. In light of this, compared with (5.13) and (5.22), *Assumption 5.2* is removed for the adaptive law with leakage method (5.11) at the expense of possibly impairing the steady-state performance of the tracking error. This will be illustrated in the simulation of adaptive control of leakage approach in Section 5.5.

5.5 Examples

In this section, two examples are used to demonstrate the effectiveness of the proposed robust adaptive tracking control scheme. One is a numerical example and the other involves an air handling unit.

5.5.1 Numerical example

Consider the following uncertain switched linear system:

$$A_1 = \begin{bmatrix} -0.6 & 3.0 & 3.3 \\ 1.0 & -0.1 & 2.1 \\ -0.2 & 2.3 & 1.5 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} -2.3 \\ 1.8 \\ 0.4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2.6 & 3.6 & 1.2 \\ 1.8 & -0.5 & 3.6 \\ 1.2 & 1.8 & 2.0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.0 \\ -0.4 \\ -1.5 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3.5 & 2.4 & 1.3 \\ 2.4 & 2.2 & 2.3 \\ 3.9 & 2.6 & -0.9 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0.2 \\ -1.3 \\ -0.8 \end{bmatrix}$$

and the following reference switched model:

$$A_{\mathrm{m}1} = \begin{bmatrix} 7.9 & 26.1 & 32.8 \\ -5.7 & -18.1 & -21.0 \\ -1.7 & -1.7 & -3.6 \end{bmatrix}, \qquad B_{\mathrm{m}1} = \begin{bmatrix} -2.3 \\ 1.8 \\ 0.4 \end{bmatrix}$$

$$A_{\mathrm{m}2} = \begin{bmatrix} 25.9 & 23.1 & 22.7 \\ -7.5 & -8.3 & -5.0 \\ -33.8 & -27.4 & -30.3 \end{bmatrix}, \qquad B_{\mathrm{m}2} = \begin{bmatrix} 0.0 \\ -0.4 \\ -1.5 \end{bmatrix}$$

$$A_{\mathrm{m}3} = \begin{bmatrix} 7.3 & 4.7 & 2.5 \\ -22.5 & -12.7 & -5.5 \\ -11.4 & -6.5 & -5.7 \end{bmatrix}, \qquad B_{\mathrm{m}3} = \begin{bmatrix} 0.2 \\ -1.3 \\ -0.8 \end{bmatrix}.$$

Let $\kappa_1 = 0.1$, $\kappa_2 = 0.15$, and $\kappa_3 = 0.12$, $K = 1$, and $h = 2$. Solving the LMIs (5.6) gives

$$P_{10} = \begin{bmatrix} 0.02 & 0.03 & 0.03 \\ 0.03 & 0.04 & 0.04 \\ 0.03 & 0.04 & 0.05 \end{bmatrix}, \qquad P_{11} = \begin{bmatrix} 0.15 & 0.17 & 0.19 \\ 0.17 & 0.30 & 0.34 \\ 0.19 & 0.34 & 0.44 \end{bmatrix}$$

$$P_{20} = \begin{bmatrix} 0.11 & 0.08 & 0.07 \\ 0.08 & 0.07 & 0.05 \\ 0.07 & 0.05 & 0.05 \end{bmatrix}, \qquad P_{21} = \begin{bmatrix} 0.70 & 0.55 & 0.46 \\ 0.55 & 0.49 & 0.36 \\ 0.46 & 0.36 & 0.34 \end{bmatrix}$$

$$P_{30} = \begin{bmatrix} 0.07 & 0.03 & 0.01 \\ 0.03 & 0.012 & 0.003 \\ 0.009 & 0.003 & 0.006 \end{bmatrix}, \quad P_{31} = \begin{bmatrix} 0.60 & 0.31 & 0.17 \\ 0.31 & 0.17 & 0.11 \\ 0.17 & 0.11 & 0.10 \end{bmatrix}.$$

We select the switching interval $t_{i+1} - t_i = \tau_{\mathrm{d}}$ for $i = 1,2,3$. Therefore, the time-varying positive matrices $P_p(t)$ for $p \in \{1,2,3\}$ are $P_p(t) = (t - \tau_{\mathrm{d}} \cdot \mathrm{floor}(t/\tau_{\mathrm{d}})) \cdot \left(P_{p,1} - P_{p,0}\right)/\tau_{\mathrm{d}} + P_{p,0}$, where floor$(t/\tau_\mathrm{d})$ rounds t/τ_d to the nearest integer less than or equal to t/τ_d.

Before performing the adaptation process, we select a bounded external disturbance $d(t) = [0.2\sin(10t)\ \ e^{-0.1t}\ \ 0.1\cos(5t)]^T$, the initial conditions $x(0) = [0\ 0\ 0]^T$, $x_\mathrm{m}(0) = [3\ 1\ 0]^T$, and the adaptive gains $\Gamma_p = 10I_3$, $\gamma_p = 10$, $\forall p \in \mathcal{M}$. The switching signal is designed with a dwell time $\tau_\mathrm{d} = 2$ as shown in Fig. 5.4.

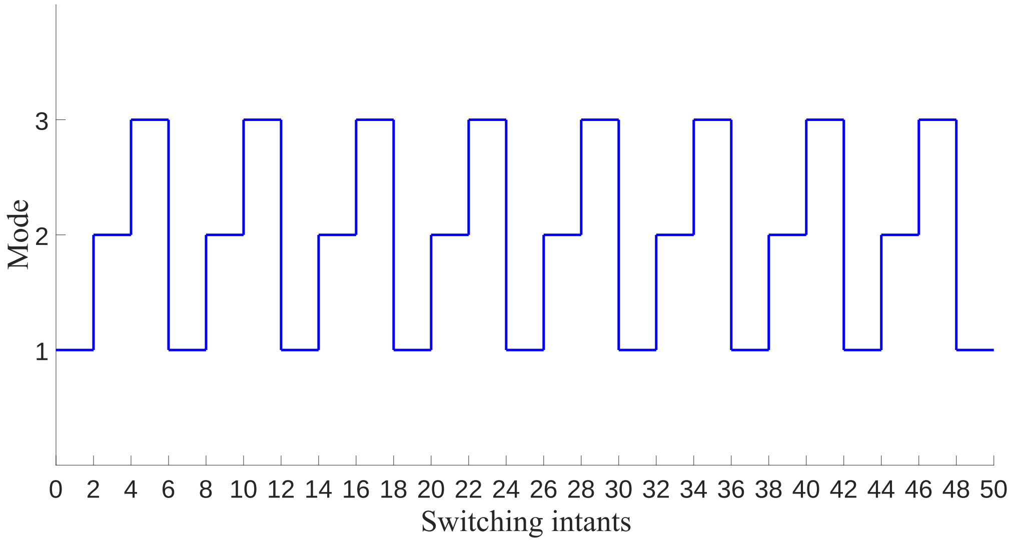

Figure 5.4: The switching signal.

A. Adaptive control with projection law

We select the initial parameter estimates $k_p(0) = 0.2k_p^*$, $l_p(0) = 0.2l_p^*$, $\forall p \in \mathcal{M}$. Assume the parameter estimates reside in the following known bounds: $k_1(t) \in \left[1.2k_1^*\ 0.2k_1^*\right]$, $k_s(t) \in \left[0.2k_s^*\ 1.2k_s^*\right]$ with $s \in \{2,3\}$, and $l_p(t) \in \left[0.2l_p^*\ 1.2l_p^*\right]$ with $p \in \{1,2,3\}$. The resulting tracking error is given in Fig. 5.5, which shows that the tracking error is attracted inside a ball.

B. Adaptive control with leakage law

The leakage rates $\delta_p^k = \delta_p^l = 0.015$ are chosen to satisfy the conditions (5.12). To study the effect of the initial conditions of the parameter estimates on the steady-state performance of the tracking error, we select the two initial conditions of the parameter estimates $k_p(0) = 0.8k_p^*$, $l_p(0) = 0.8l_p^*$, and $k_p(0) = 0.2k_p^*$, $l_p(0) = 0.2l_p^*$, $\forall p \in \mathcal{M}$. The resulting tracking errors based on the two initial conditions of parameter estimates are given in Figs. 5.6–5.7, which show that the tracking errors are attracted inside a ball. By comparing Fig. 5.6 and Fig. 5.7, it can be observed that larger initial parameter estimation errors give rise to larger tracking errors. Moreover, it can be noticed from Fig. 5.5 and Fig. 5.7 that the ultimate bound of tracking error $\mathscr{B}_\mathrm{leak}$ in Fig. 5.7 is bigger than $\mathscr{B}_\mathrm{proj}$ in Fig. 5.5, which indicates that the adaptive law with leakage may negatively impact the steady-state performance of the tracking error when improper initial conditions in (5.8)–(5.11) are selected.

5.5.2 Air handling unit

We consider an air handling unit (AHU) serving a conditioned space [47, 93]. For the sake of simplicity, the space is represented by a single zone and the ducting is omitted. The air han-

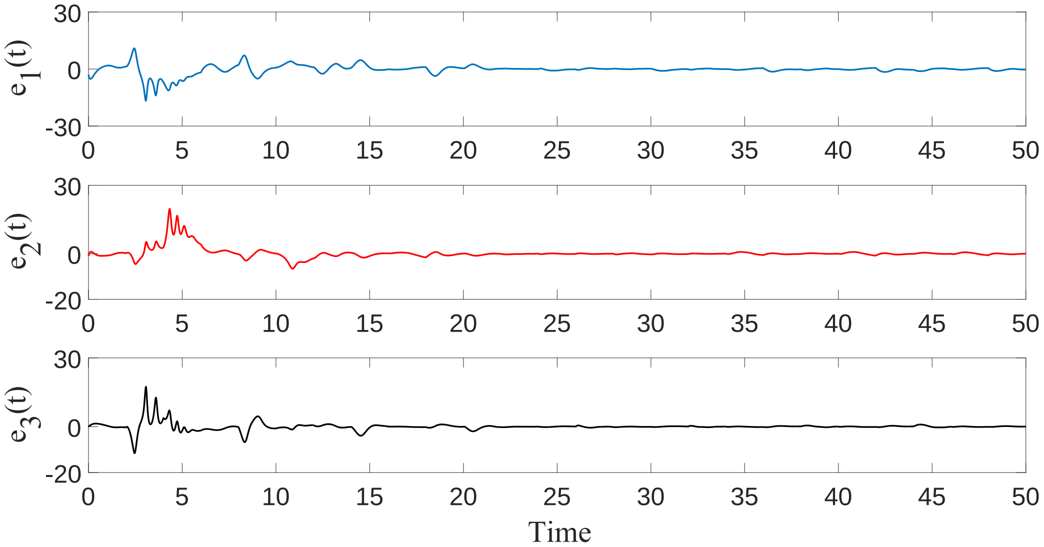

Figure 5.5: The tracking error $e(t)$ via projection laws.

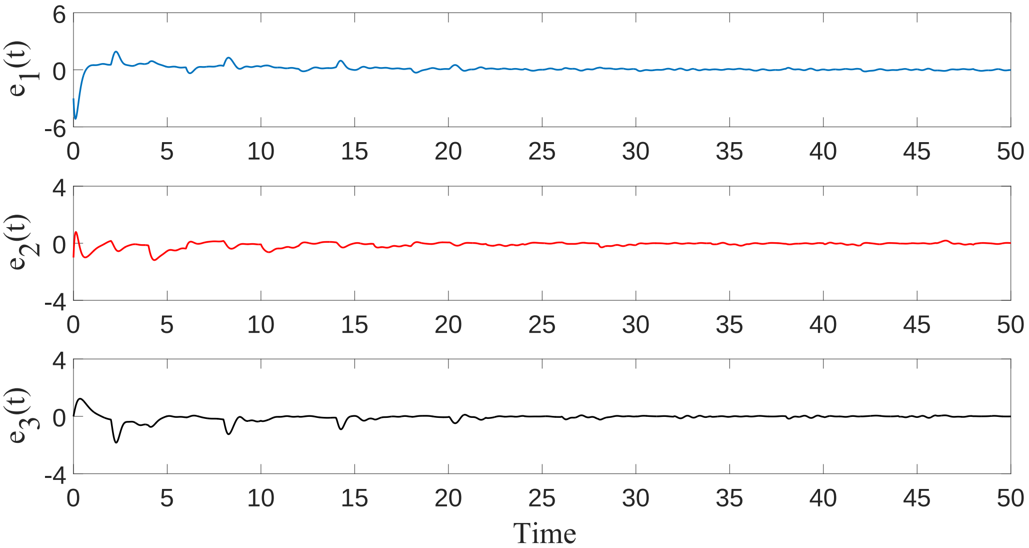

Figure 5.6: The tracking error $e(t)$ with $k_p(0) = 0.8k_p^*$ and $l_p(0) = 0.8l_p^*$.

dler consists of a cooling/heating coil, supply and return fans, valve, damper, and ductwork, as shown in Fig. 5.8. Chilled or heated water flows through the coil depending on different seasons, and the water flow rate is controlled by a programmable valve. The air in the ductwork is cooled or heated when flowing through the coil, and the air flow rate is controlled by the speed of the supply/return fan. In addition to the supply air, the temperature in the zone is affected by the heat exchange with the building mass, and by solar radiation, which has both a direct effect (through windows) and an indirect effect (through the building mass) on the zone temperature.

All these effects can be derived from the heat balance of the zone. In particular, after ignoring the influence of humidity on temperature, the following differential equations de-

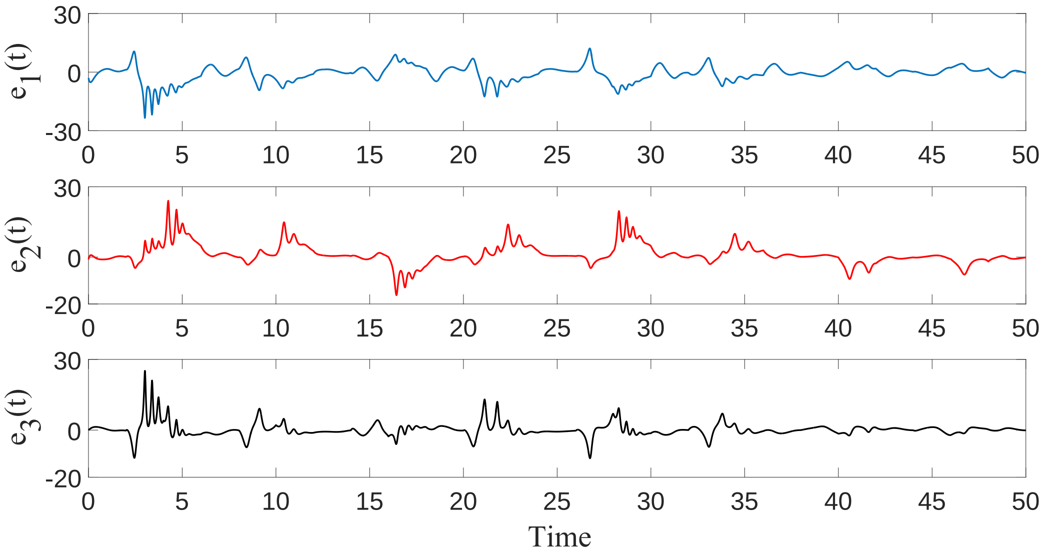

Figure 5.7: The tracking error $e(t)$ with $k_p(0) = 0.2k_p^*$ and $l_p(0) = 0.2l_p^*$.

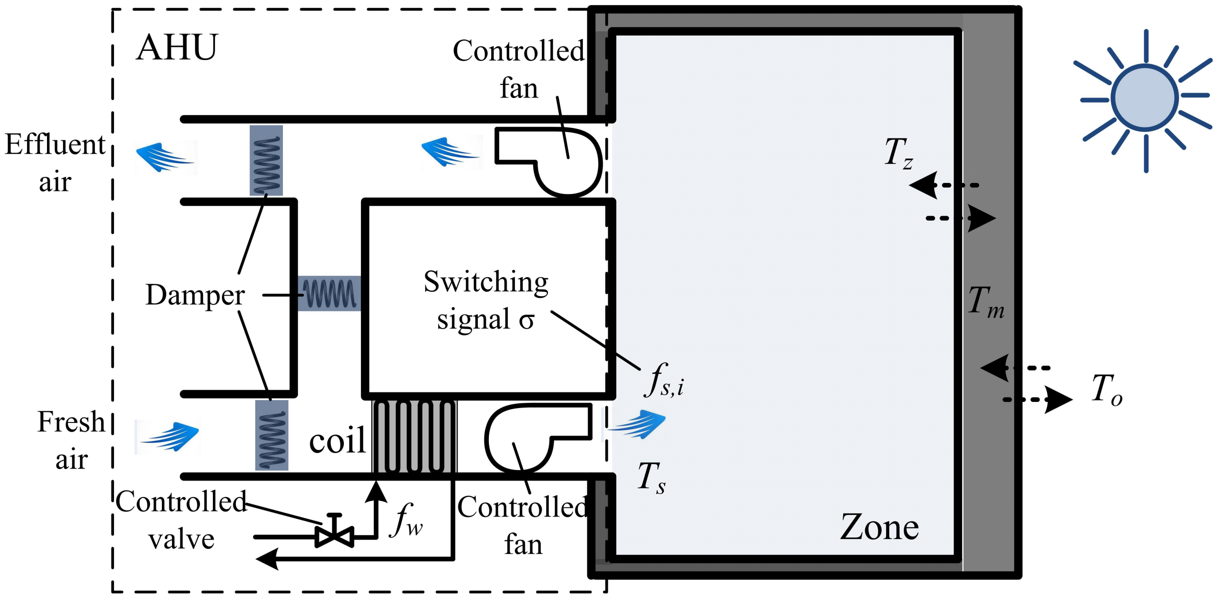

Figure 5.8: Layout of the AHU with a single zone.

scribe the evolution of the temperature in the system:

$$
\begin{aligned}
\dot{T}_\mathrm{z} &= -\frac{f_{\mathrm{s},i}}{V_\mathrm{z}}(T_\mathrm{z} - T_\mathrm{s}) + \frac{k_\mathrm{mz}}{\rho_\mathrm{a} C_\mathrm{pa} V_\mathrm{z}}(T_\mathrm{m} - T_\mathrm{z}) + \frac{\eta P_\mathrm{solar}}{\rho_\mathrm{a} C_\mathrm{pa} V_\mathrm{z}} \\
\dot{T}_\mathrm{m} &= \frac{k_\mathrm{mz}}{C_\mathrm{m}}(T_\mathrm{z} - T_\mathrm{m}) + \frac{k_\mathrm{om}}{C_\mathrm{m}}(T_\mathrm{o} - T_\mathrm{m}) + \frac{1-\eta}{C_\mathrm{m}} P_\mathrm{solar} \\
\dot{T}_\mathrm{s} &= \frac{f_{\mathrm{s},i}}{V_\mathrm{c}}(T_\mathrm{z} - T_\mathrm{s}) + \frac{0.25 f_{\mathrm{s},i}}{V_\mathrm{c}}(T_\mathrm{o} - T_\mathrm{z}) + \frac{\rho_\mathrm{w} C_\mathrm{pw} \Delta T_\mathrm{c}}{\rho_\mathrm{a} C_\mathrm{pa} V_\mathrm{c}} f_\mathrm{w}
\end{aligned}
\tag{5.27}
$$

where T_s, T_m, and $T_\mathrm{z} \in \mathbb{R}$ represent the temperature of the supply air, of the mass, and of the thermal zone, respectively; the coefficient 0.25 represents the typical damper position where 25% fresh air and 75% exhaust air are mixed in the mixing zone before passing through the coil. The definitions of all the parameters of the dynamic model (5.27) are provided in Table 5.1. The following comments apply to the volumetric flow rate of air $f_{s,i}$. In most building automation designs, the variable-speed drive of the fan operates at fixed number of speeds [40], typically $\{$*off, low, medium, high*$\} = \{0,1,2,3\}$, giving $f_{s,i} \in \{f_{\mathrm{s},0}, f_{\mathrm{s},1}, f_{\mathrm{s},2}, f_{\mathrm{s},3}\}$ with $f_{\mathrm{s},0} \equiv 0$. The switching behavior between different fan speeds, i.e. $f_{s,i} = f_{s,\sigma(t)}$, is controlled by the switching signal $\sigma(\cdot)$, which might depend on external commands or by a supervisory controller.

Table 5.1: Parameters of the dynamical model (5.27).

C_{pa} (J/kg°C)	Specific heat capacity of air
C_{pw} (J/kg°C)	Specific heat capacity of water
C_{m} (J/°C)	Heat capacity of the mass
k_{om} (W/°C)	Thermal conductance between environmental air and the mass
k_{mz} (W/°C)	Thermal conductance between indoor air and the mass
ρ_{a} (kg/m^3)	Air density
ρ_{w} (kg/m^3)	Water density
V_{c} (m^3)	Volume of heat exchanger
V_{z} (m^3)	Volume of thermal space
T_{o} (°C)	Environmental temperature
T_{s} (°C)	Temperature of supply air
T_{z} (°C)	Temperature of thermal zone
T_{m} (°C)	Temperature of mass in the zone
f_{s} (m^3/s)	Volumetric flow rate of air with discrete modes
f_{w} (m^3/s)	Flow rate of chilled water
ΔT_{c} (°C)	Temperature gradient in cooling unit
P_{solar} (kW/m^2)	Solar radiation

By taking into account the practical constraints of $f_{\mathrm{s},i}$ and f_{w}, and after selecting the state $x = [T_{\mathrm{z}}\ T_{\mathrm{m}}\ T_{\mathrm{s}}]^T$, the model (5.27) is recast into in a switched input-saturated system with four different modes (subsystems)

$$\dot{x}(t) = A_i x(t) + b f_{\mathrm{w}}(t) + d(t), \quad i \in \{0,1,2,3\} \tag{5.28}$$

where

$$A_i = \begin{bmatrix} -\alpha_1 f_{\mathrm{s},i} - \alpha_2 k_{\mathrm{mz}} & -\alpha_2 k_{\mathrm{mz}} & \alpha_1 f_{\mathrm{s},i} \\ \beta_1 k_{\mathrm{mz}} & -\beta_1 (k_{\mathrm{mz}} + k_{\mathrm{om}}) & 0 \\ 0.75\beta_2 f_{\mathrm{s},i} & 0 & -\beta_2 f_{\mathrm{s},i} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \gamma_1 \end{bmatrix}, \quad d = d_1 + d_2, \quad d_1 = \begin{bmatrix} 0 \\ 0 \\ \gamma_2 \end{bmatrix} f_{\mathrm{s},i}, \quad d_2 = \begin{bmatrix} \xi_1 \\ \xi_2 \\ 0 \end{bmatrix}$$

Table 5.2: The parameters of the AHU.

C_{pa}	1000 J/kg
C_{pw}	4180 J/kg°C
C_{m}	880 J/kg°C
k_{om}	$5.235 \cdot 10^{-5}$ W/°C
k_{mz}	$2.141 \cdot 10^{-4}$ W/°C
ρ_{a}	1.18 kg/m^3
ρ_{w}	1000 kg/m^3
V_{c}	1 m^3
V_{z}	400 m^3
T_{o}	0 °C
f_{s}	2.6 m^3/s, 5 m^3/s, 7 m^3/s
ΔT_{c}	6 °C
P_{solar}	(700W/m^2)

with

$$\alpha_1 = \frac{1}{V_{\mathrm{z}}}, \quad \alpha_2 = \frac{1}{\rho_{\mathrm{a}} C_{\mathrm{pa}} V_{\mathrm{z}}}, \quad \beta_1 = \frac{1}{C_{\mathrm{m}}}, \quad \beta_2 = \frac{1}{V_{\mathrm{c}}}, \quad \gamma_1 = \frac{\rho_{\mathrm{w}} C_{\mathrm{pw}} \Delta T_{\mathrm{c}}}{\rho_{\mathrm{a}} C_{\mathrm{pa}} V_{\mathrm{c}}}$$

$$\gamma_2 = 0.25\beta_2 T_{\mathrm{o}}, \quad \xi_1 = \frac{\eta}{\rho_{\mathrm{a}} C_{\mathrm{pa}} V_{\mathrm{z}}} P_{\mathrm{solar}}, \quad \xi_2 = \frac{1-\eta}{C_{\mathrm{m}}} P_{\mathrm{solar}} + \beta_1 k_{\mathrm{om}} T_{\mathrm{o}}.$$

Since the switched system (5.28) is uncontrollable with $f_{s,0} \equiv 0$, in what follows, we consider three modes $f_{\mathrm{s},i}$, $i = 1,2,3$ to develop the switching signal. The following parameters are adopted to study the air handling unit. The material of the building mass is burnt brick, whose density is 1820 kg/m^3, specific heat is 880 J/kg°C, and absorptivity of the solar radiation is 0.6. The discretized error of the water supply rate is 0.00005m^3/s and the water supply rate is constrained in [0 0.0025]m^3/s. The desired zone temperature is 24°C, and the initial zone temperature is 16°C. The thermal and geometrical parameters are given in Table 5.2[1].

A. Design of reference models

The discrete fan speeds $\{f_{\mathrm{s},1}, f_{\mathrm{s},2}, f_{\mathrm{s},3}\}$ lead to a simple design of x_{m}: that is, the dynamics of the reference model are designed to be faster or slower according to the fan speed. In particular, the higher the speed, the faster the rate with which the temperature is expected to change. Then, the following reference models for three different fan speeds are designed:

$$A_{\mathrm{m},1} = \begin{bmatrix} -0.0066 & -0.0001 & 0.0065 \\ 0.0002 & -0.0003 & 0 \\ 3.2 & 0 & -4.4 \end{bmatrix}, \quad b_{\mathrm{m},1} = \begin{bmatrix} 0 \\ 0 \\ 1.2638 \end{bmatrix}$$

[1] The solar radiation absorbed by the building depends on several factors, like a) design variables (geometrical features of building like orientation); b) material of the building mass; c) environmental weather data like different seasons, etc. This gives rise to $P_{\mathrm{solar}} \in$ [0 700]W/m^2. In the example, we take the maximum value, i.e. $P_{\mathrm{solar}} = 700$W/m^2.

$$A_{\mathrm{m},2} = \begin{bmatrix} -0.0126 & -0.0001 & 0.0125 \\ 0.0002 & -0.0003 & 0 \\ 3 & 0 & -4.1 \end{bmatrix}, \quad b_{\mathrm{m},2} = \begin{bmatrix} 0 \\ 0 \\ 1.1310 \end{bmatrix}$$

$$A_{\mathrm{m},3} = \begin{bmatrix} -0.0201 & -0.0001 & 0.02 \\ 0.0002 & -0.0003 & 0 \\ 2.5 & 0 & -3.6 \end{bmatrix}, \quad b_{\mathrm{m},3} = \begin{bmatrix} 0 \\ 0 \\ 1.1169 \end{bmatrix}$$

whose step responses are shown in Fig. 5.9. From Fig. 5.9, we learn that for the low speed

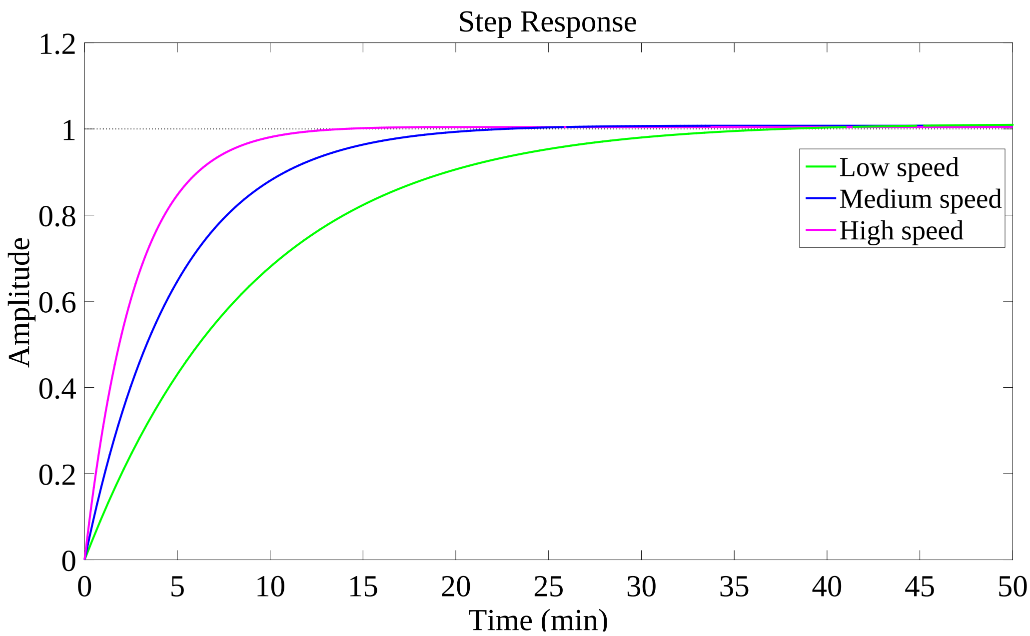

Figure 5.9: The step response of the dynamic of the zone temperature.

of the fan, it takes 36 minutes to reach the steady state; for the medium speed of the fan, it takes 22 minutes to reach the steady state; for the high speed of the fan, it takes 16 minutes to reach the steady state.

Solving the LMIs (5.6) for $K = 1$ using the SeDuMi solver, we obtain the dwell time $\tau_\mathrm{d} = 1.2s$ and the matrices

$$P_{1,0} = \begin{bmatrix} 2.9958 & 0.1765 & -0.0525 \\ 0.1765 & 3.4025 & 0.0003 \\ -0.0525 & 0.0003 & 0.0790 \end{bmatrix}, \quad P_{1,1} = \begin{bmatrix} 3.1765 & 0.1658 & -0.1387 \\ 0.1658 & 3.4572 & 0.0002 \\ -0.1387 & 0.0002 & 0.2020 \end{bmatrix}$$

$$P_{2,0} = \begin{bmatrix} 2.7996 & 0.1694 & -0.0496 \\ 0.1694 & 3.4025 & 0.0005 \\ -0.0496 & 0.0005 & 0.0800 \end{bmatrix}, \quad P_{2,1} = \begin{bmatrix} 3.3691 & 0.1724 & -0.1366 \\ 0.1724 & 3.4572 & 0.0005 \\ -0.1366 & 0.0005 & 0.2103 \end{bmatrix}$$

$$P_{3,0} = \begin{bmatrix} 2.6126 & 0.1615 & -0.0423 \\ 0.1615 & 3.4025 & 0.0009 \\ -0.0423 & 0.0009 & 0.0819 \end{bmatrix}, \quad P_{3,1} = \begin{bmatrix} 3.5182 & 0.1777 & -0.1291 \\ 0.1777 & 3.4572 & 0.0012 \\ -0.1291 & 0.0012 & 0.2357 \end{bmatrix}.$$

B. Adaptive control of air handling unit

Let the initial conditions be $x(0) = [16\ 5\ 16]^T$, $x_{\mathrm{m}}(0) = [16.5\ 5\ 16.5]^T$, $k_1(0) = [0.2941\ 0\ -0.4234]^T \times 10^{-4}$, $k_2(0) = [-0.1764\ 0\ 0.2117]^T \times 10^{-4}$, $k_3(0) = [-0.823\ 0\ 1.035]^T \times 10^{-4}$, $l_1(0) = 2.9731 \times 10^{-5}$, $l_2(0) = 2.6606 \times 10^{-5}$, $l_3(0) = 2.6606 \times 10^{-5}$, $v(0) = 4.7049 \times 10^{-8}$. In addition, the adaptive gains $\Gamma_i = 10^{-5} I$ and $\gamma_{1,i} = \gamma_{2,i} = \gamma_{3,i} = 10^{-5}$ for $i = 1,2,3$ are chosen. We assume that the mode of the fan is switched every 10 minutes in 60 minutes. The switching law is given in Fig. 5.10.

Then, the dynamics of the zone temperature and the dynamics of reference zone temperature with the switching signal in Fig. 5.10 are shown in Fig. 5.11, which indicates an ultimate bounded tracking error. This result is consistent with the proposed result as (5.13).

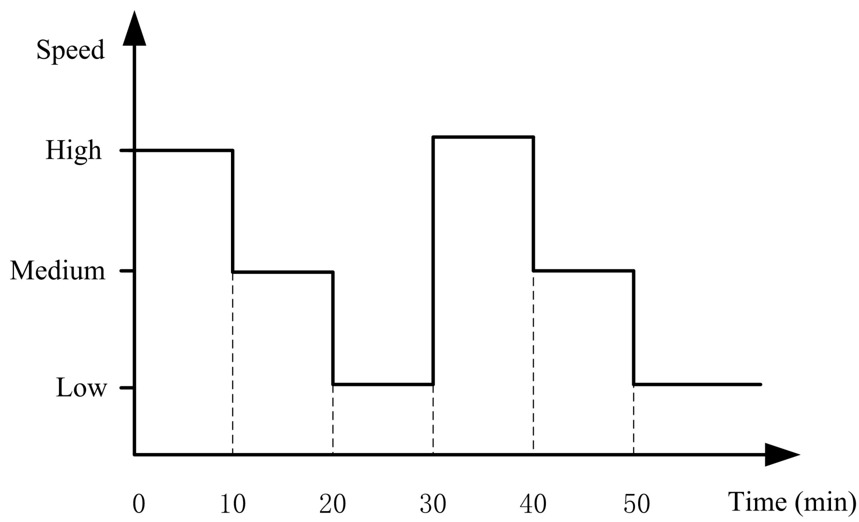

Figure 5.10: The time-driven switching signal.

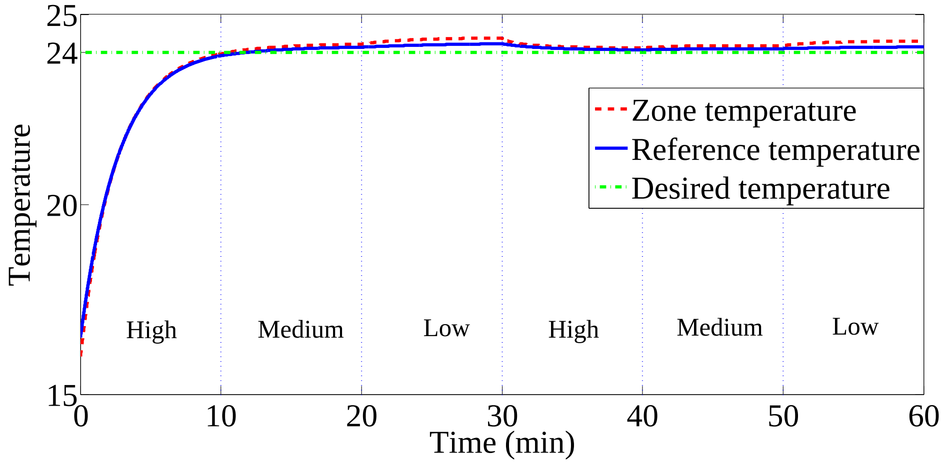

Figure 5.11: The dynamics of the zone temperature and the reference zone temperature with switching signal as in Fig. 5.10.

5.6 Concluding remarks

Robust adaptive control problem of uncertain switched linear systems has been studied in this chapter. As an extension of the results in Chapter 4, two control schemes have been introduced based on a parameter projection approach and a leakage approach, respectively. With the proposed robust adaptive control schemes, the closed-loop switched linear systems have been shown to be globally uniformly ultimately bounded. In addition, ultimate bounds of the tracking error for both cases have been given, which has indicated that as a price paid for not requiring the bounds of the unknown parameters, the leakage approach tends to produce worse steady-state performance compared with parameter projection. One numerical example and one practical example involving an air handling unit are used to show the effectiveness of the robust adaptive controllers.

Part II

Adaptive and Robust Stabilization of Switched Linear Systems with Delays

Chapter 6

Adaptive Stabilization of Switched Linear Systems with Time-Varying Delays

In the presence of discontinuous time-varying delays of states, neither Krasovskii nor Razumikhin techniques can be successfully applied to adaptive stabilization of switched time-delay systems with parametric uncertainties. This chapter develops a new adaptive control scheme for uncertain switched time-delay systems that can handle impulsive behavior in both states and time-varying delays. At the core of the proposed scheme is a Lyapunov function with a dynamically time-varying coefficient, which allows the Lyapunov function to be non-increasing at the switching instants. The control scheme can guarantee global uniform ultimate boundedness of the adaptive closed-loop system.

6.1 Introduction

Switched time-delay systems are natural generalizations of switched systems, as time delay of states is another common problem in hybrid systems. Time delay is typically time-varying, and makes the state of a system evolve based on some delayed information [32, 98, 105, 113, 146]. Stability and stabilization of switched time-delay systems have been studied intensively [3, 17, 66, 85, 98, 128]. However, the two main approaches adopted to deal with time-varying delay, namely the Krasovskii technique and the Razumikhin technique, show some limitations when applied to adaptive stabilization of switched time-delay systems. Since the Krasovskii technique involves the bounded derivatives of the time-varying delays, continuity of time-varying delay at the switching instants should be assumed [58, 84, 98]. While this assumption might be reasonable for non-switched systems, it turns out to be quite restrictive when considering that switching behavior may lead to impulsive delays. On the other hand, even although the Razumikhin technique can handle discontinuous time-varying delays, its application in an adaptive stabilization setting is problematic: as pointed out in [81, 82, 156], the selection of the Razumikhin coefficient is limited to an unknown interval inside which the existence of an adaptive controller is guaranteed. Therefore, addressing discontinuous time-varying delays in adaptive control of uncertain switched systems is not only practically relevant but it also tackles the need to extend the current stabilization tools, which motivates this study.

In this chapter, we develop a new adaptive control design for uncertain switched linear systems that can handle impulses in both states and time-varying delays. A stability condition is developed to deal with the discontinuities of multiple time-varying delays. Based on the stability condition, a new adaptive controller is proposed by solving a family of Riccati equations and LMIs. The adaptive law involves a piecewise dynamic gain that is properly designed to guarantee the non-increasing property of the Lyapunov function at the switching instants. Furthermore, a less conservative switching law based on mode-dependent dwell time is designed by exploiting information of each subsystem. With the designed adaptive controller and switching law, global uniform ultimate boundedness of the closed-loop system can be guaranteed. The main contribution of this chapter is that the impulsive behavior of both the states and the time-varying delay is addressed and solved for the first time in an adaptive stabilization setting. As a matter of fact, the proposed adaptive mechanism substantially enlarges the class of uncertain switched linear systems for which the adaptive stabilization can be solved.

This chapter is organized as follows: the problem formulation and some useful lemmas are given in Section 6.2. A stability condition for switched linear systems with time-varying delays is introduced in Section 6.3. In Section 6.4, the adaptive controller with matching conditions is designed while the adaptive controller with unmatched uncertainties is designed in Section 6.5. A two-tank system is used to illustrate the proposed method in Section 6.6. The chapter is concluded in Section 6.7.

6.2 Problem statement

Consider the switched linear impulsive system with multiple time-varying delays

$$\begin{aligned}\dot{x}(t) &= \big(A_{\sigma(t)} + \Delta A_{\sigma(t)}(t)\big)\, x(t) + B_{\sigma(t)} u(t) + w(t)\\ &\quad + \sum_{\ell=1}^{L} \big(E_{\ell,\sigma(t)} + \Delta E_{\ell,\sigma(t)}(t)\big)\, x(t - d_{\ell,\sigma(t)}(t))\\ x_{t_0}(\vartheta) &= \psi(\vartheta), \quad \vartheta \in [t_0 - d_{\mathrm{m}}, t_0]\\ x(t_i) &= H_{\sigma(t)} x(t_i^-), \quad i \in \mathbb{N}^+\end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the system input, $w \in \mathbb{R}^n$ is a bounded disturbance with unknown bound $\bar{w}$, i.e., $\|w\| \le \bar{w}$. The matrices $A_p \in \mathbb{R}^{n\times n}$, $E_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, and $H_p \in \mathbb{R}^{n\times n}$ are known constant matrices with (A_p, B_p), being controllable for all $p \in \mathcal{M}$; $\Delta A_p \in \mathbb{R}^{n\times n}$ and $\Delta E_{\ell,p} \in \mathbb{R}^{n\times n}$ are unknown possibly time-varying matrices. The terms $d_{\ell,p}(\cdot) \in \mathbb{R}$, $\ell \in \mathcal{L}$, $p \in \mathcal{M}$, represent unknown multiple time-varying delays, and $\psi(\vartheta)$ is a continuous initial function for $\vartheta \in [t_0 - d_{\mathrm{m}}, t_0]$ with d_{m} defined in Assumption 6.1. The switching signal $\sigma : [0,\infty) \to \mathcal{M} := \{1,2,\ldots,M\}$ is a piecewise function with M denoting the number of subsystems.

The following assumptions are made.

Assumption 6.1 *There exists a positive constant $d_{\mathrm{m}} =: \sup_{\ell\in\mathcal{L},p\in\mathcal{M}} d_{\ell,p}(\cdot)$, which is not necessarily known.*

Assumption 6.2 *The uncertain matrices $\Delta A_p(\cdot)$ and $\Delta E_p(\cdot)$ satisfy the following matching conditions:*

$$\Delta A_p(t) = B_p \Xi_p(t), \quad \Delta E_{\ell,p}(t) = B_p \Pi_{\ell,p}(t) \tag{6.2}$$

with $\|\Xi_p(t)\|^2 \le \xi_p$ and $\|\Pi_{\ell,p}(t)\|^2 \le \zeta_{\ell,p}$ where ξ_p and $\zeta_{\ell,p}$, $p \in \mathcal{M}$, $l \in \mathcal{L}$, are unknown positive constants.

Remark 6.1 Assumption 6.1 only requires the existence of an upper bound to the multiple time-varying delays. Note that the time-varying delays are allowed be discontinuous at the switching instants due to switching behavior. Discontinuity excludes the application of the Krasovskii technique, while the Razumikhin technique is intrinsically subject to limitations in the adaptive control setting, as highlighted in [81, 156]. Therefore, a new stability condition needs to be developed for adaptive control of system (6.1). Assumption 6.2 is rather standard, and widely used in adaptive control or robust control [44, 156] to dominate the uncertainties. Note that Assumption 6.2 will be relaxed in (6.23), so as to handle bounded unmatched uncertainties. □

The following lemmas are useful for deriving the main results.

Lemma 6.1 *[129] Let $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, and M, N be appropriately dimensioned constant matrices. Then, for any positive constant ϵ, it holds that*

$$2y^T MNz \le \epsilon y^T MM^T y + \epsilon^{-1} z^T N^T Nz.$$

Lemma 6.2 *[17] For given positive scalars $\mu \ge 1$, a, and b that satisfy $0 < b < a\mu/(\mu+1)$, define*

$$v =: \frac{1}{c} \operatorname{arctanh} \frac{(\mu-1)c}{\frac{a}{2}(\mu+1) - 2b} \tag{6.3}$$

where $c = \sqrt{a^2/4 - b^2/\mu}$. Let $\varphi(t)$ be the solution of the following initial value problem:

$$\begin{aligned} \dot{\varphi}(t) &= -\frac{v}{T}\left(\varphi^2(t) - a\varphi(t) + \frac{b^2}{\mu}\right), \quad t \ge t_s \\ \varphi(t_s) &= \frac{b}{\mu} \end{aligned} \tag{6.4}$$

with $T > 0$. Then, $\varphi(t)$ exists on $[t_s, \infty)$ and satisfies

$$\varphi(t) = \frac{\frac{a}{2} + c + (\frac{a}{2} - c)\varpi(t)}{1 + \varpi(t)}, \quad t \ge t_s \tag{6.5}$$

where $\varpi(t) = \frac{a/2 + c - b/\mu}{b/\mu - a/2 + c} e^{-\frac{2cv}{T}(t - t_s)}$, $\varphi(t_s + T) = b$, and $\dot{\varphi}(t) \ge 0$.

6.3 Stability analysis

In this section, a new control scheme is proposed based on the solution of a family of LMIs and Riccati equations to guarantee global uniform ultimate boundedness of the closed-loop system. The following lemma extends the results of [81] to switched systems with impulsive behavior, which is crucial to derive the stability results.

Lemma 6.3 *Let $g(\cdot)$ be a left-continuous function for all $t \ge t_0$ and $\phi(\cdot) > 0$ be continuous for*

$t \in [t_0 - d_\mathrm{m}, t_0]$. *If there exist positive constants* α_1, α_2, α_3 *with* $\alpha_1 > \alpha_2$ *such that*

$$\begin{aligned}
&\dot{g}(t) \le -\alpha_1 g(t) + \alpha_2 \sup_{t-d_\mathrm{m}\le s\le t} g(s) + \alpha_3,\ t \in [t_i, t_{i+1})\\
&g(t_{i+1}^-) \ge g(t_{i+1})\\
&g_{t_0}(\vartheta) = \phi(\vartheta), \quad \vartheta \in [t_0 - d_\mathrm{m}, t_0]
\end{aligned}$$

then we have

$$g(t) \le \beta_1 + \beta_2 e^{-\rho(t-t_0)}, \quad t \ge t_0$$

where $\beta_1 = \alpha_3/(\alpha_1 - \alpha_2)$, $\beta_2 = \sup_{t_0-d_\mathrm{m}\le s\le t_0} \phi(s) - \beta_1$, *and* ρ *is the unique solution to* $\rho = \alpha_1 - \alpha_2 e^{\rho d_\mathrm{m}}$.

Proof: To facilitate the proof, consider the differential equation

$$\begin{aligned}
&\dot{f}(t) = -\alpha_1 f(t) + \alpha_2 \sup_{t-d_\mathrm{m}\le s\le t} f(s) + \alpha_3, \quad t \ge t_0\\
&f_{t_0}(\vartheta) = \phi(\vartheta), \quad \vartheta \in [t_0 - d_\mathrm{m}, t_0].
\end{aligned} \tag{6.6}$$

We search for a unique solution to (6.6) in the form

$$f(t) = \beta_1 + \beta_2 e^{-\rho(t-t_0)}, \quad t \ge t_0 \tag{6.7}$$

with $\beta_1 > 0$, β_2, $\rho > 0$ to be determined, which implies that $\sup_{t-d_\mathrm{m}\le s\le t} f(s) = f(t - d_\mathrm{m})$. Note that uniqueness of (6.7) arises from continuity of the right-hand side of the differential equation. Substituting (6.7) into (6.6) leads to

$$\begin{aligned}
-\rho\beta_2 e^{-\rho(t-t_0)} = &-\alpha_1\beta_1 + \alpha_2\beta_1 + \alpha_3\\
&-\alpha_1\beta_2 e^{-\rho(t-t_0)} + \alpha_2\beta_2 e^{-\rho(t-t_0-d_\mathrm{m})}
\end{aligned}$$

which gives the solutions to β_1, β_2 and the characteristic equation of ρ

$$\beta_1 = \frac{\alpha_3}{\alpha_1 - \alpha_2}, \quad \beta_2 = f_{t_0} - \beta_1, \quad \rho = \alpha_1 - \alpha_2 e^{\rho d_\mathrm{m}}$$

where a solution to ρ always exists and is unique due to $\alpha_1 \ge \alpha_2$, and $\beta_2 \le \sup_{t_0-d_\mathrm{m}\le s\le t_0} \phi(s) - \beta_1$. It can be verified that $g(t) \le f(t)$ for $t \in [t_i, t_{i+1})$. Considering $g(t_{i+1}^-) \ge g(t_{i+1})$ at the switching instant t_{i+1}, we arrive at $g(t_{i+1}) \le f(t_{i+1})$. This implies, together with (6.7)

$$g(t) \le \frac{\alpha_3}{\alpha_1 - \alpha_2} + \beta_2 e^{-\rho(t-t_0)}, \quad t \ge t_0$$

where

$$\beta_2 = \sup_{t_0-d_\mathrm{m}\le s\le t_0} \phi(s) - \frac{\alpha_3}{\alpha_1 - \alpha_2}.$$

This completes the proof. □

6.4 Adaptive control design with matched uncertainties

Now we are ready to present the stability result using Lemmas 6.1–6.3.

Theorem 6.2 *Suppose that there exist a collection of symmetric positive definite matrices P_p, Q_p, $G_p \in \mathbb{R}^{n\times n}$, positive scalars a, b, v, χ τ_p, $\mu \geq 1$, $\varepsilon_{\ell,p}$, $\epsilon_{\ell,p}$, $\varrho_{1,p}$, $\varrho_{2,p}$, $\ell \in \mathcal{L}$, $p \in \mathcal{M}$, such that $b < a\mu/(\mu+1)$, v satisfies* (6.3)*, and*

$$\begin{bmatrix} \Psi_p & P_pE_{1,p} & \cdots & P_pE_{L,p} \\ * & -\varepsilon_{1,p}^{-1}I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & -\varepsilon_{L,p}^{-1}I \end{bmatrix} < 0 \tag{6.8a}$$

$$\begin{bmatrix} -\frac{v}{\tau_p}P_p & -G_p \\ * & -\frac{b^2 v}{\mu\tau_p}P_p \end{bmatrix} < 0 \tag{6.8b}$$

$$\chi P_p - b\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1}\right)I > 0 \tag{6.8c}$$

$$H_q^T P_q H_q \leq \mu P_p \tag{6.8d}$$

with

$$\Psi_p = -Q_p + \frac{v}{\tau_p}aP_p + \chi P_p + 2G_p$$

where P_p and Q_p, $\varrho_{1,p}$, $\varrho_{2,p}$, $p \in \mathcal{M}$, and κ satisfy the following Riccati equation

$$A_p^T P_p + P_pA_p + \left(\varrho_{1,p}^{-1} + \varrho_{2,p}^{-1}\right)I - 2\kappa P_pB_pB_p^TP_p = -Q_p. \tag{6.9}$$

Then, under Assumptions 6.1 and 6.2, the controller

$$u(t) = -\left(\kappa + \frac{1}{2}\hat{\theta}(t)\right)B_{\sigma(t)}^T P_{\sigma(t)}x(t) \tag{6.10}$$

and the adaptive law

$$\dot{\hat{\theta}}(t) = \gamma\varphi_{\mathrm{m}}(t)x^T(t)P_{\sigma(t)}B_{\sigma(t)}B_{\sigma(t)}^TP_{\sigma(t)}x(t) - \gamma\delta\hat{\theta}(t) \tag{6.11}$$

with $\gamma > 0$ a given adaptive gain, $\delta \geq \chi/\gamma$, and

$$\varphi_{\mathrm{m}}(t) = \begin{cases} \varphi(t), & t \in [t_i, t_i + \tau_{\sigma(t)}) \\ b, & t \in [t_i + \tau_{\sigma(t)}, t_{i+1}) \\ \frac{b}{\mu}, & t = t_{i+1} \end{cases} \tag{6.12}$$

and $\varphi(\cdot)$ as in (6.5) *with $T = \tau_{\sigma(t)}$ and $t_s = t_i$ guarantees that the switched impulsive system* (6.1) *is GUUB for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_p)$. Moreover, an ultimate bound for the norm of the state is given by*

$$b_{\mathrm{T}} = \sqrt{\frac{b\mu\bar{w}^2\max_{p\in\mathcal{M}}\varrho_{2,p}\overline{\lambda} + \frac{\delta}{2}\theta^2}{b\underline{\lambda}\chi - b\mu\max_{p\in\mathcal{M}}\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1}\right)}} \tag{6.13}$$

where $\underline{\lambda} \triangleq \min_{p\in\mathcal{M}} \lambda_{\min}(P_p)$, $\overline{\lambda} \triangleq \max_{p\in\mathcal{M}} \lambda_{\max}(P_p)$, *and*

$$\theta \triangleq \max_{p\in\mathcal{M}} \left\{ \xi_p \varrho_{1,p} + \sum_{\ell=1}^{L} \zeta_{\ell,p} \epsilon_{\ell,p} \right\}. \tag{6.14}$$

Proof: In this proof, the time index is sometimes not indicated for compactness, and a delayed signal will be marked with the subscript d, e.g. $x_{\mathrm{d}} = x(t - d_{\ell,p}(t))$. Consider the following Lyapunov function:

$$V(t) = \varphi_{\mathrm{m}}(t) x^T(t) P_{\sigma(t)} x(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t) \tag{6.15}$$

with $\tilde{\theta} = \theta - \hat{\theta}$. It is straightforward that $V(\cdot)$ is continuous during the switching intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}^+$, and possibly discontinuous at the switching instants t_i, $i \in \mathbb{N}^+$. Without loss of generality, we assume that subsystem p is active for $t \in [t_i, t_{i+1})$ and subsystem q is active for $t \in [t_{i+1}, t_{i+2})$. Moreover, to facilitate the analysis of the Lyapunov function, we partition the interval $[t_i, t_{i+1})$ into two parts: $[t_i, t_i + \tau_p)$ and $[t_i + \tau_p, t_{i+1})$, upon which, according to (6.12), (6.15) can be recast into

$$V(t) = \begin{cases} \varphi(t) x^T(t) P_p x(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t), & t \in [t_i, t_i + \tau_p) \\ b x^T(t) P_p x(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t), & t \in [t_i + \tau_p, t_{i+1}). \end{cases}$$

The essence of the proof is to show that the Lyapunov function satisfies the conditions in Lemma 6.3. The proof is organized in three steps:

(a) for $t \in [t_i, t_i + \tau_p)$, the Lyapunov function is shown to satisfy the conditions in Lemma 6.3 using the LMIs (6.8a)–(6.8c), the Riccati equation (6.9), and the adaptive controller (6.10)–(6.12);

(b) for $t \in [t_i + \tau_p, t_{i+1})$, the Lyapunov function is shown to satisfy the conditions in Lemma 6.3 using the LMIs (6.8a) and (6.8c), the Riccati equation (6.9), and the adaptive controller (6.10)–(6.12);

(c) at the switching instant t_{i+1}, the Lyapunov function is shown to be non-increasing due to (6.8d) and the reset of $\varphi_{\mathrm{m}}(t_{i+1})$.

(a) For $t \in [t_i, t_i + \tau_p)$, it can be shown that the time derivative of $V(\cdot)$ is

$$\begin{aligned} \dot{V} \le{}& \varphi x^T \left(A_p^T P_p + P_p A_p + \sum_{\ell=1}^{L} \varepsilon_{\ell,p} P_p E_{\ell,p} E_{\ell,p}^T P_p \right) x \\ &+ \varphi x^T \left(\sum_{\ell=1}^{L} \epsilon_{\ell,p} P_p \Delta E_{\ell,p} \Delta E_{\ell,p}^T P_p \right) x + \varrho_{1,p}^{-1} \varphi x^T x \\ &+ \varrho_{1,p} \varphi x^T P_p \Delta A_p \Delta A_p^T P_p x + 2\varphi x^T P_p B_p u \\ &+ \sum_{\ell=1}^{L} \left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1} \right) \varphi x_{\mathrm{d}}^T x_{\mathrm{d}} + \varrho_{2,p} \varphi w^T P_p P_p w \\ &+ \varrho_{2,p}^{-1} \varphi x^T x - \frac{v}{\tau_p} \left(\varphi^2 - a\varphi + \frac{b^2}{\mu} \right) x^T P_p x - \frac{1}{\gamma} \tilde{\theta} \dot{\hat{\theta}} \end{aligned} \tag{6.16}$$

where the inequality holds according to Lemma 6.1 and Lemma 6.2. Using Assumption 6.1 and the fact that $\varphi > 0$, (6.16) is written as

$$\begin{aligned}\dot{V} \leq{}& \varphi x^T \left(A_p^T P_p + P_p A_p + \sum_{\ell=1}^{L} \varepsilon_{\ell,p} P_p E_{\ell,p} E_{\ell,p}^T P_p \right) x \\ &+ \left(\xi_p \varrho_{1,p} + \sum_{\ell=1}^{L} \zeta_{\ell,p} \epsilon_{\ell,p} \right) \varphi x^T P_p B_p B_p^T P_p x \\ &+ 2\varphi x^T P_p B_p u + \left(\varrho_{1,p}^{-1} + \varrho_{2,p}^{-1} \right) \varphi x^T x \\ &- \frac{\upsilon}{\tau_p} \left(\varphi^2 - a\varphi + \frac{b^2}{\mu} \right) x^T P_p x - \frac{1}{\gamma} \tilde{\theta} \dot{\hat{\theta}} \\ &+ \sum_{\ell=1}^{L} \left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1} \right) \varphi x_{\mathrm{d}}^T x_{\mathrm{d}} + \varrho_{2,p} \varphi w^T P_p P_p w.\end{aligned} \tag{6.17}$$

Then, substituting the Riccati equation (6.9) into (6.17) yields

$$\begin{aligned}\dot{V} \leq{}& -\varphi x^T Q_p x + 2\kappa \varphi x^T P_p B_p B_p^T P_p x \\ &+ \varphi x^T \left(\sum_{\ell=1}^{L} \varepsilon_{\ell,p} P_p E_{\ell,p} E_{\ell,p}^T P_p \right) x \\ &+ \left(\xi_p \varrho_{1,p} + \sum_{\ell=1}^{L} \zeta_{\ell,p} \epsilon_{\ell,p} \right) \varphi x^T P_p B_p B_p^T P_p x \\ &+ 2\varphi x^T P_p B_p u + \varrho_{2,p} \varphi w^T P_p P_p w \\ &- \frac{\upsilon}{\tau_p} \left(\varphi^2 - a\varphi + \frac{b^2}{\mu} \right) x^T P_p x - \frac{1}{\gamma} \tilde{\theta} \dot{\hat{\theta}} \\ &+ \sum_{\ell=1}^{L} \left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1} \right) \varphi x_{\mathrm{d}}^T x_{\mathrm{d}}.\end{aligned}$$

With help of the controller (6.10), the adaptive law (6.11), and the definition of θ in (6.14), one has

$$\begin{aligned}\dot{V} \leq{}& -\varphi x^T Q_p x + \varphi x^T \left(\sum_{\ell=1}^{L} \varepsilon_{\ell,p} P_p E_{\ell,p} E_{\ell,p}^T P_p \right) x \\ &+ \sum_{\ell=1}^{L} \left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1} \right) \varphi x_{\mathrm{d}}^T x_{\mathrm{d}} + \delta \tilde{\theta} \hat{\theta} \\ &- \frac{\upsilon}{\tau_p} \left(\varphi^2 - a\varphi + \frac{b^2}{\mu} \right) x^T P_p x \\ &+ \varrho_{2,p} \varphi w^T P_p P_p w.\end{aligned}$$

Furthermore, (6.8b) directly shows that

$$\begin{bmatrix} \varphi x \\ x \end{bmatrix}^T \begin{bmatrix} -\frac{\upsilon}{\tau_p} P_p & -G_p \\ * & -\frac{b^2 \upsilon}{\mu \tau_p} P_p \end{bmatrix} \begin{bmatrix} \varphi x \\ x \end{bmatrix} < 0$$

which, combined with (6.8a) by Schur complement, implies

$$\begin{aligned}\dot V \le &-\chi\varphi x^T P_p x + \varrho_{2,p}\varphi w^T P_p P_p w\\ &+\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1}+\epsilon_{\ell,p}^{-1}\right)\varphi x_{\mathrm d}^T x_{\mathrm d} + \delta\tilde\theta\hat\theta.\end{aligned}$$

Recalling that $\tilde\theta = \theta - \hat\theta$ and using $\delta\tilde\theta\theta - \delta\tilde\theta^2 \le -\frac{1}{2}\delta\tilde\theta^2 + \frac{1}{2}\delta\theta^2$ results in

$$\begin{aligned}\dot V \le &-\chi\varphi x^T P_p x - \frac{\chi}{2\gamma}\tilde\theta^2 + \varrho_{2,p}\varphi w^T P_p P_p w\\ &+\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1}+\epsilon_{\ell,p}^{-1}\right)\varphi x_{\mathrm d}^T x_{\mathrm d} + \frac{1}{2}\left(\frac{\chi}{\gamma}-\delta\right)\tilde\theta^2 + \frac{\delta}{2}\theta^2\end{aligned}$$

where $\chi/\gamma - \delta \le 0$. In addition, the following holds:

$$\begin{aligned}\varphi x_{\mathrm d}^T x_{\mathrm d} &\le \frac{\mu}{\lambda_{\min}(P_p)}\varphi_{\mathrm d} x_{\mathrm d}^T P_p x_{\mathrm d}\\ &\le \frac{\mu}{\lambda_{\min}(P_p)} V_{\mathrm d}\\ &\le \frac{\mu}{\lambda_{\min}(P_p)} \sup_{t-d_{\mathrm m}\le s\le t} V(s).\end{aligned} \tag{6.18}$$

Hence, the derivative of V for $t \in [t_i, t_i + \tau_p)$ satisfies

$$\begin{aligned}\dot V \le &-\chi V + \frac{\delta}{2}\theta^2 + \varrho_{2,p}\varphi w^T P_p P_p w\\ &+\frac{\mu\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1}+\epsilon_{\ell,p}^{-1}\right)}{\lambda_{\min}(P_p)} \sup_{t-d_{\mathrm m}\le s\le t} V(s).\end{aligned} \tag{6.19}$$

(b) For $t \in [t_i + \tau_p, t_{i+1})$, the Lyapunov function becomes

$$V(t) = b x^T(t) P_{\sigma(t)} x(t) + \frac{1}{2\gamma}\tilde\theta^2(t).$$

It follows immediately from (6.8a) that

$$\begin{bmatrix}\Theta_p & P_pE_{1,p} & \cdots & P_pE_{L,p}\\ * & -\varepsilon_{1,p}^{-1}I & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & -\varepsilon_{L,p}^{-1}I\end{bmatrix} < 0$$

with $\Theta_p = -Q_p + \chi P_p$, which, combined with (6.8c) and following the similar steps from (6.16) to (6.19) yields

$$\begin{aligned}\dot V \le &-\chi V + \frac{\delta}{2}\theta^2 + \varrho_{2,p} b w^T P_p P_p w\\ &+\frac{\mu\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1}+\epsilon_{\ell,p}^{-1}\right)}{\lambda_{\min}(P_p)} \sup_{t-d_{\mathrm m}\le s\le t} V(s).\end{aligned} \tag{6.20}$$

According to (6.19) and (6.20), it holds for $t \in [t_i, t_{i+1})$ that

$$\begin{aligned} \dot{V} \leq & -\chi V + \frac{\delta}{2}\theta^2 + \varrho_{2,p}\varphi_{\mathrm{m}} w^T P_p P_p w \\ & + \frac{\mu \sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1}\right)}{\lambda_{\min}(P_p)} \sup_{t-d_{\mathrm{m}} \leq s \leq t} V(s). \end{aligned} \tag{6.21}$$

(c) At the switching instant t_{i+1}, using (6.8d) and the fact that $\varphi(t_{i+1}^-) = b$ and $\varphi(t_{i+1}) = \frac{b}{\mu}$, one has

$$\begin{aligned} & V(t_{i+1}) - V(t_{i+1}^-) \\ & = \varphi(t_{i+1}) x^T(t_{i+1}) P_q x(t_{i+1}) - b(t_{i+1}^-) x^T(t_{i+1}^-) P_p x(t_{i+1}^-) \\ & = \frac{b}{\mu} x^T(t_{i+1}^-) H_q^T P_q H_q x(t_{i+1}^-) - b x^T(t_{i+1}^-) P_p x(t_{i+1}^-) \\ & = b x^T(t_{i+1}^-)\left(\frac{H_q^T P_q H_q}{\mu} - P_p\right) x(t_{i+1}^-) \\ & \leq 0 \end{aligned} \tag{6.22}$$

which implies that (6.21) holds for all $t \geq t_0$. In light of this, using (6.8c) and Lemma 6.3, it readily follows that

$$V(t) \leq \frac{b\bar{w}^2 \max_{p\in\mathcal{M}} \varrho_{2,p}\lambda_{\max}^2(P_p) + \frac{\delta}{2}\theta^2}{\chi - \frac{\mu\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1}\right)}{\min_{p\in\mathcal{M}}\lambda_{\min}(P_p)}} + \beta_2 e^{-\rho(t-t_0)}$$

where β_2 is a finite constant dependent on the initial value of the Lyapunov function. This indicates, together with (6.15), the ultimate bound b_{T} shown in (6.13). This completes the proof. □

Remark 6.3 Some comments are needed for the family of Riccati equations (6.9). Since (A_p, B_p) is controllable for all $p \in \mathcal{M}$, one can always find a solution for P_p and Q_p satisfying (6.9). As a matter of fact, the Riccati equations guarantee a sufficient large stability margin with only the requirement of controllability. In [156], a LMI condition is proposed to design the adaptive controller for time-varying delay without considering switching behavior of the system: however, the absence of a Riccati equation fundamentally requires the system matrix A_p to be Hurwitz, which to a large extent limits the scope of applications of the method in [156]. □

Remark 6.4 In contrast with the Razumikhin technique, where an adaptive controller is guaranteed to exist only in an unknown interval, the existence of the adaptive controller (6.10)–(6.12) is well defined by the appropriate selection of the constants in Theorem 6.2. Here are some guidelines for the selection of such constants: after a sufficiently large stability margin has been achieved by the solution of the Riccati equations (6.9), one can find a feasible μ in (6.8d); at this point, with a simple grid search over the pair (a, b) (which automatically defines ν from (6.3)), we have that (6.8a)–(6.8c) are linear in the decision variables G_p, τ_p^{-1}, $\varepsilon_{\ell,p}^{-1}$, $\epsilon_{\ell,p}^{-1}$. One can either solve a feasibility problem, or preferably, optimize the solution to the LMIs for large τ_p^{-1} (to address a desirable large family of switching laws), or large $\epsilon_{\ell,p}^{-1}$ and $b\underline{\lambda}\chi - b\mu \max_{p\in\mathcal{M}} \sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1}\right)$ (to minimize the ultimate bound b_{T} in (6.13)). □

Remark 6.5 Different from classic adaptive laws with a constant gain, the proposed design incorporates a piecewise dynamic gain φ_{m}, entering both the adaptive law (6.11) and the Lyapunov function (6.15). Note that (6.22) suggests that the Lyapunov function (6.15) is non-increasing at switching instants thanks to the dynamic gain φ_{m} in (6.12). It can be verified that in the absence of disturbances and time-varying delays, asymptotic stability of the closed-loop system can be derived. In fact, using the controller (6.10) and the adaptive law (6.11) with $\delta \equiv 0$, (6.21) reduces to $\dot{V} \leq -\chi\varphi_{\mathrm{m}} x^T P_p x$, and using Barbalat's lemma [48] leads to asymptotic stability. This implies, in the spirit of [135], that the Lyapunov function (6.15) can lead to less conservative result than using standard multiple Lyapunov functions [91], i.e., with $\varphi_{\mathrm{m}} \equiv 1$. □

Remark 6.6 Connected to the previous remark, a question may arise: why cannot the time-interpolation method of [135] (which is also based on a Lyapunov function non-increasing at the switching instants) be adopted to achieve the control objective of this chapter? Some clarifications are provided as follows: instead of using a constant P_p for each subsystem, [135] relies on a time-varying $P_p(t)$, $t \in [t_i, t_{i+1})$, obtained by linear interpolation of a set of positive definite matrices (c.f. Lemma 1 in [135]). However, the need for the Riccati equations in (6.9), which are quadratic in P_p, makes linear interpolation not applicable here. □

6.5 Adaptive control design with unmatched uncertainties

In many practical cases, the uncertainties may not satisfy the matching conditions shown in (6.2). For the unmatched case, Assumption 6.2 can be relaxed into Assumption 6.3.

Assumption 6.3 *The uncertain matrices $\Delta A_p(\cdot)$ and $\Delta E_{\ell,p}(\cdot)$ satisfy*

$$\begin{aligned} \Delta A_p(t) &= B_p \Xi_p(t) + \Delta\Xi_p(t) \\ \Delta E_{\ell,p}(t) &= B_p \Pi_{\ell,p}(t) + \Delta\Pi_{\ell,p}(t) \end{aligned} \tag{6.23}$$

with $\|\Xi_p(t)\|^2 \leq \xi_p$, $\|\Delta\Xi_p(t)\|^2 \leq \Delta\xi_p$, and $\|\Pi_{\ell,p}(t)\|^2 \leq \zeta_{\ell,p}$, and $\|\Delta\Pi_{\ell,p}(t)\|^2 \leq \Delta\zeta_{\ell,p}$, $p \in \mathcal{M}$, $\ell \in \mathcal{L}$, where ξ_p and $\zeta_{\ell,p}$ are unknown positive constants, and $\Delta\xi_p$ and $\Delta\zeta_{\ell,p}$ are known positive constants.

Remark 6.7 Since the unmatched uncertainties (6.23) cannot be addressed by the controller in an adaptive fashion, the knowledge of the bounds of the unmatched uncertainties is required to guarantee stability of the switched system. In fact, to the best of the author' knowledge, how to cope with unknown unmatched uncertainties without knowing their bounds is still an open problem in adaptive control or robust control [44]. □

Considering the unmatched terms as in (6.23), we provide the following stability result:

Corollary 6.1 *Suppose that there exist a collection of positive definite symmetric matrices P_p, Q_p, $G_p \in \mathbb{R}^{n\times n}$, positive scalars a, b, v, χ τ_p, $\mu \geq 1$, $\varepsilon_{\ell,p}$, $\epsilon_{\ell,p}$, $\iota_{\ell,p}$, $\varrho_{1,p}$, $\varrho_{2,p}$, $\ell \in \mathcal{L}$, $p \in \mathcal{M}$ such* *that* $b < a\mu/(\mu+1)$, v *satisfies* (6.3), *and*

$$\begin{bmatrix} \Psi_p & P_pE_{1,p} & \cdots & P_pE_{L,p} & \sqrt{\Delta\zeta_{1,p}}P_p & \cdots & \sqrt{\Delta\zeta_{L,p}}P_p \\ * & -\varepsilon_{1,p}^{-1}I & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & -\varepsilon_{L,p}^{-1}I & 0 & \cdots & 0 \\ * & * & \cdots & * & -\iota_{1,p}^{-1}I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & * & \cdots & -\iota_{L,p}^{-1}I \end{bmatrix} < 0$$

$$\begin{bmatrix} -\frac{v}{\tau_p}P_p & -G_p \\ * & -\frac{b^2v}{\mu\tau_p}P_p \end{bmatrix} < 0$$

$$\chi P_p - b\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1} + \iota_{\ell,p}^{-1}\right)I > 0$$

$$H_q^TP_qH_q \le \mu P_p$$

with

$$\Psi_p = -Q_p + \frac{v}{\tau_p}aP_p + \chi P_p + 2G_p$$

where P_p *and* Q_p, $\varrho_{1,p}$, $\varrho_{2,p}$, $p \in \mathcal{M}$, *and* κ *satisfy the Riccati equation*

$$\begin{aligned}\left(A_p + \sqrt{\Delta\xi_p}I\right)^T P_p + P_p\left(A_p + \sqrt{\Delta\xi_p}I\right)& \\ + \left(\varrho_{1,p}^{-1} + \varrho_{2,p}^{-1}\right)I - 2\kappa P_pB_pB_p^TP_p &= -Q_p.\end{aligned}$$

Then, under Assumptions 6.1 and 6.3, the controller

$$u(t) = -\left(\kappa + \frac{1}{2}\hat{\theta}(t)\right)B_{\sigma(t)}^TP_{\sigma(t)}x(t)$$

and the adaptive law

$$\dot{\hat{\theta}}(t) = \gamma\varphi_{\mathrm{m}}(t)x^T(t)P_{\sigma(t)}B_{\sigma(t)}B_{\sigma(t)}^TP_{\sigma(t)}x(t) - \gamma\delta\hat{\theta}(t)$$

with $\gamma > 0$ *a given adaptive gain,* $\delta \ge \chi/\gamma$, *and* φ_{m} *as defined in* (6.12) *guarantee that the switched impulsive system* (6.1) *is GUUB for any switching signal* $\sigma(\cdot) \in \mathcal{D}(\tau_p)$. *Moreover, an ultimate bound for the norm of the state is given by*

$$b_{\mathrm{T}} = \sqrt{\frac{b\mu\bar{w}^2\max_{p\in\mathcal{M}}\varrho_{2,p}\overline{\lambda} + \frac{\delta}{2}\theta^2}{b\underline{\lambda}\chi - b\mu\max_{p\in\mathcal{M}}\sum_{\ell=1}^{L}\left(\varepsilon_{\ell,p}^{-1} + \epsilon_{\ell,p}^{-1} + \iota_{\ell,p}^{-1}\right)}}$$

where $\underline{\lambda} \triangleq \min_{p\in\mathcal{M}} \lambda_{\min}(P_p)$, $\overline{\lambda} \triangleq \max_{p\in\mathcal{M}} \lambda_{\max}(P_p)$, *and*

$$\theta \triangleq \max_{p\in\mathcal{M}} \left\{ \xi_p \varrho_{1,p} + \sum_{\ell=1}^{L} \zeta_{\ell,p} \epsilon_{\ell,p} \right\}.$$

Proof: The proof follows similar steps as the proof of Theorem 6.2, and thus it is omitted. □

6.6 Example

Consider the two-tank system taken from [6, 18], and illustrated in Fig. 6.1. The states of the system are the deviations of reservoir levels with respect to their nominal values, denoted by the dashed lines in Fig. 6.1. The flow between the two reservoirs is proportional to the difference of their levels. We assume that both flow control and level measurement can switch

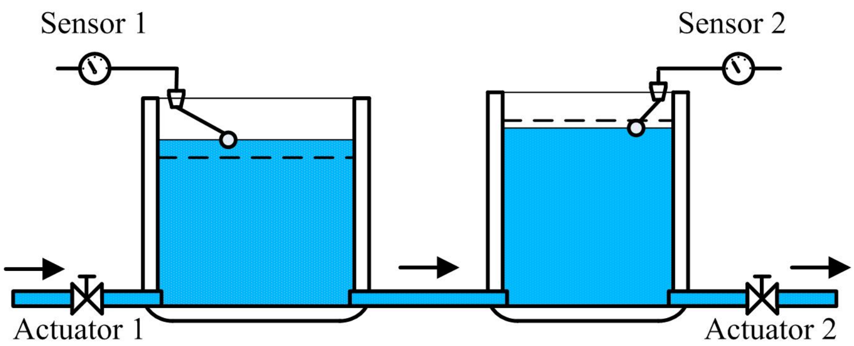

Figure 6.1: The two-tank system.

between the first tank (actuator 1-sensor 1) and the second tank (actuator 2-sensor 2). In addition, the pipe between the two tanks gives rise to time delays. Thus, the two-tank system can be modeled as an impulsive switched system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \left(E_{\sigma(t)} + \Delta E_{\sigma(t)}\right) x(t - d_{\sigma(t)}(t)) + B_{\sigma(t)} u(t) + w(t) \\ x(t_i) &= H_{\sigma(t)} x(t_i^-) \end{aligned}$$

where the following matrices have been taken in line with [6, 18]:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 & -0.2 \\ 0.2 & 0.4 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.2 & -0.3 \\ -0.2 & 0.4 \end{bmatrix}$$

$$\Delta E_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0 \end{bmatrix}, \quad \Delta E_2 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.95 & 0 \\ 0 & 1.05 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 1.05 & 0 \\ 0 & 0.95 \end{bmatrix}$$

and $d_1(t) = 0.1(1-\cos(t))$, $d_2(t) = 0.1(1+\sin(t))$, and $w(t) = 0.1\cos(5t)$. Let $\varrho_{11} = \varrho_{12} = \varrho_{21} = \varrho_{22} = 0.1$, $\epsilon_1 = \epsilon_2 = 1000$, $\kappa = 10$, $a = 10$, $b = 2$, and $\chi = 0.25$, $\gamma = 1$, $\delta = 0.3$, and

$$Q_1 = \begin{bmatrix} 8 & 1.9 \\ 1.9 & 10 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 2.5 & 0.9 \\ 0.9 & 3 \end{bmatrix}.$$

Solving the Riccati equations (6.9) and the LMIs (6.8a)–(6.8d) results in

$$P_1 = \begin{bmatrix} 0.8661 & 0.6171 \\ 0.6171 & 3.7130 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.9884 & 0.3496 \\ 0.3496 & 0.5164 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0.0414 & 0.0066 \\ 0.0066 & 0.0227 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.0184 & 0.0062 \\ 0.0062 & 0.0099 \end{bmatrix}$$

the mode-dependent dwell time $\tau_1 = 1.25$, $\tau_2 = 3$, and $\mu = 11.76$. For simulations, the following initial conditions are selected: $x_0 = [1\ 1.5]^T$, $\theta(0) = 1$. To illustrate the effect of the dynamic gain $\varphi_{\rm m}$ on the Lyapunov function V, we use the function $V_{\rm m} = \varphi_{\rm m} x^T P_p x$. Based on the switching signal in Fig. 6.3, the evolution of $V_{\rm m}$ is given in Fig. 6.3, which shows that $V_{\rm m}$ and thus V is decreasing at the switching instant[1]. In addition, the state response in Fig. 6.4 admits an ultimate bound 0.25, as expected from the GUUB result of Theorem 6.2.

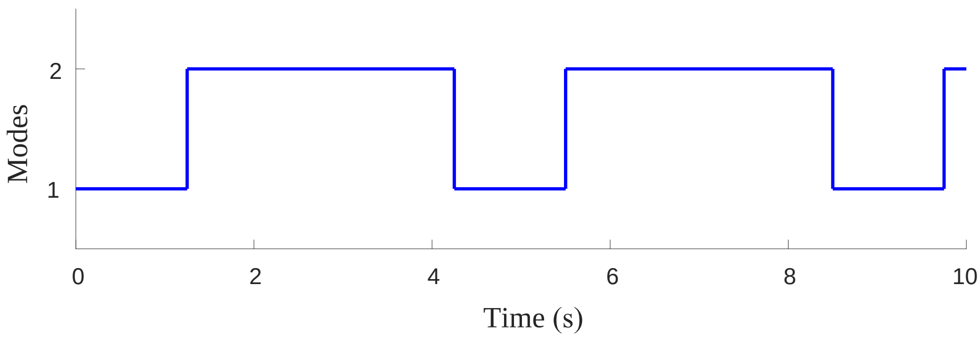

Figure 6.2: The switching law.

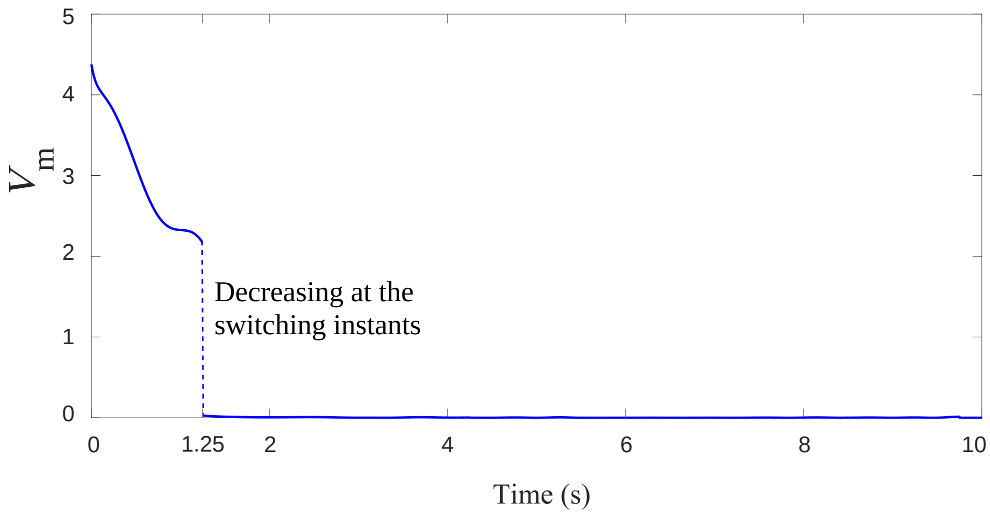

Figure 6.3: The evolution of $V_{\rm m}$.

[1] Since the signal $\tilde{\theta}$ is unknown and continuous for all $t \geq t_0$, the absence of the quadratic term $\tilde{\theta}$ in $V_{\rm m}$ does not impact the non-increasing effect of $\varphi_{\rm m}$.

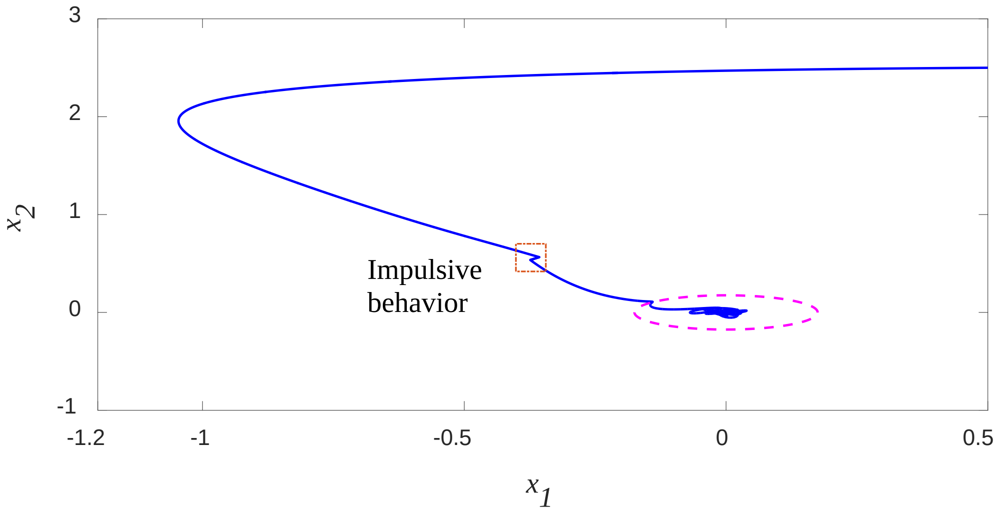

Figure 6.4: The state trajectory and the attraction region.

6.7 Concluding remarks

This chapter has investigated adaptive stabilization of switched impulsive systems with possibly discontinuous time-varying delays. By solving a family of Riccati equations and LMIs, a novel adaptive controller and a less conservative switching law based on mode-dependent dwell time than that based on dwell time have been designed. A piecewise dynamic gain has been designed for the adaptive law, which allows the Lyapunov function to be non-increasing at the switching instants. Based on the proposed control scheme, global uniform ultimate boundedness of the closed-loop systems has been guaranteed. A two-tank system has been used to illustrate the effectiveness of the control scheme.

Chapter 7

Stability and Robust Stabilization of Switched Linear Systems with Switching Delays

This chapter proposes a novel Lyapunov function to study switched linear systems with switching delays between activation of system modes and activation of candidate controller modes. The novelty consists in continuity of the Lyapunov function at the switching instants and discontinuity when the system modes and controller modes are matched. This structure is exploited to construct a time-varying Lyapunov function that is non-increasing at time instants of discontinuity. Stability criteria based on the novel Lyapunov function are developed to guarantee global asymptotic stability in the noiseless case. Most importantly, when exogenous disturbances are considered, the proposed Lyapunov function can be used to guarantee a finite non-weighted $\mathcal{L}_2$ gain for switched systems with switching delays, for which Lyapunov functions proposed in literature are inconclusive. A numerical example illustrates the effectiveness of the proposed method.

The research presented in this chapter has been published in [138].

7.1 Introduction

Typically, the focus of stability and stabilization of switched linear systems is on switched linear systems without switching delays, an ideal case in which the controller is assumed to switch synchronously with the system mode. However, due to delay between a mode change and activation of its corresponding controller, or due to the time needed to detect switching of system mode, nonzero time intervals, called *unmatched intervals*, are present during which system modes and controller modes are mismatched. A typical example in engineering practice can be seen in teleoperation, e.g. [74]. Most of the research on ideal switched linear systems has been carried out based on the Lyapunov function proposed by Branicky [9] that is discontinuous at the switching instants and continuous during the switching intervals between two consecutive switching instants. Two properties of the Lyapunov function have been exploited to develop switching strategies based on dwell time and average dwell time [37, 154]: an exponential decreasing during the switching interval between two consecutive switching instants, and a bounded increment of the Lyapunov function at switching instants. For switched linear systems with switching delays, several studies have appeared on stability and stabilization problems [30, 106, 121, 143, 144, 152]. In particular, a seminal work on stability of switched linear systems considering switching delays [143] introduces a new Lyapunov function for switched systems with switching delays based on the classical Lyapunov function for ideal switched systems. This new Lyapunov function uses the additional property that the Lyapunov function is allowed to increase during the unmatched intervals.

Another fundamental topic, the $\mathcal{L}_2$ gain of switched linear systems, has been intensively investigated [54, 64, 76, 98, 133, 141, 150, 155]. A weighted $\mathcal{L}_2$ gain for ideal switched linear systems based on average dwell time switching was introduced initially in [141]. Subsequently, a non-weighted $\mathcal{L}_2$ gain for ideal switched linear systems was obtained in [1] via dwell time switching laws and in [148] via switching laws using persistent dwell time. However, to the best of the author' knowledge, only a weighted $\mathcal{L}_2$ gain has been obtained for switched linear systems with switching delays [56, 57, 114, 125, 144], which is based on the Lyapunov function in [144] via average dwell time switching laws; considering the narrower class of switching laws based on dwell time does not help in achieving a non-weighted $\mathcal{L}_2$ gain[1]. In view of this, this chapter focuses on achieving a non-weighted $\mathcal{L}_2$ gain for switched linear systems with switching delays. The solution consists in designing a new Lyapunov function to cover the gap between ideal and switched linear systems with switching delays.

In this chapter, a novel Lyapunov function is proposed to study switched linear systems with switching delays; this Lyapunov function is continuous at switching instants and discontinuous at the instant when the controller and the system mode is matched. This is different with the well-known multiple Lyapunov functions proposed by Branicky [9], which are discontinuous at switching instants and continuous during the switching intervals. The major idea behind the novel Lyapunov function is the consistency with the switching mechanism of switched systems with switching delays, since the same controller is used during the matched interval of the previous subsystem and the unmatched interval of the current subsystem. The structure of the Lyapunov function is exploited to develop novel stability criteria that can be combined with the interpolation technique in [1, 12, 13, 123, 124, 126, 127] such that global asymptotic stability of switched linear systems with switching delays is guaranteed. The contribution of this chapter is twofold:

- A new Lyapunov function is proposed that is consistent with the controller design of switched linear systems with switching delays;
- A non-weighted $\mathcal{L}_2$ gain is guaranteed for the first time for switched linear systems with switching delays.

This chapter is organized as follows: Section 7.2 introduces the problem formulation and some preliminaries. Section 7.3 gives a condition in the form of linear matrix inequalities (LMIs) to guarantee global asymptotic stability for switched linear systems considering switching delays. Section 7.4 derives the LMI conditions for a non-weighted $\mathcal{L}_2$ gain and H_∞ control of switched linear systems with switching delays. A numerical example is adopted to illustrate the theoretical results in Section 7.5. The chapter is concluded in Section 7.6.

[1]This can be verified by setting $N_0 \equiv 1$ to the derivation in [125] according to the definition of dwell time [36].

7.2 Problem statement

Consider the following switched linear system:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}w(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) + F_{\sigma(t)}w(t) \end{aligned} \tag{7.1}$$

where $t \geq 0$, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^g$ is the output, $w \in \mathbb{R}^r$ is an exogenous disturbance, and the switching signal $\sigma : [0, \infty) \to \mathcal{M} := \{1, 2, \ldots, M\}$ is a piecewise function with M denoting the number of subsystems. In this chapter, a mode-dependent state-feedback controller is adopted, *i.e.*, $u(t) = G_{\sigma(t)}x(t)$. Let $\Delta\tau_i$ be the delay before switching to a new subsystem and the activation of the corresponding controller after the switching instant t_i. Then, the switched linear system (7.1) becomes a switched linear system with switching delays as follows:

$$\begin{aligned} \dot{x}(t) &= (A_{\sigma(t)} + B_{\sigma(t)}G_{\sigma(t-\Delta\tau_i)})x(t) + E_{\sigma(t)}w(t) \\ &= \begin{cases} \overline{A}_{p,q}x(t) + E_p w(t), & t \in [t_i,\, t_i + \Delta\tau_i) \\ \overline{A}_p x(t) + E_p w(t), & t \in [t_i + \Delta\tau_i,\, t_{i+1}) \end{cases} \\ y(t) &= (C_{\sigma(t)} + D_{\sigma(t)}G_{\sigma(t-\Delta\tau_i)})x(t) + F_{\sigma(t)}w(t) \\ &= \begin{cases} \overline{C}_{p,q}x(t) + F_p w(t), & t \in [t_i,\, t_i + \Delta\tau_i) \\ \overline{C}_p x(t) + F_p w(t), & t \in [t_i + \Delta\tau_i,\, t_{i+1}) \end{cases} \end{aligned} \tag{7.2}$$

where $\overline{A}_p$ is a Hurwitz matrix, and $\overline{A}_{p,q}$, $p \neq q \in \mathcal{M}$, may be an unstable matrix. To keep the notation concise, we denote the unmatched interval $[t_i,\, t_i + \Delta\tau_i)$ by $\mathcal{T}_{\uparrow}(t_i, t_{i+1})$, and the matched interval $[t_i + \Delta\tau_i,\, t_{i+1})$ by $\mathcal{T}_{\downarrow}(t_i, t_{i+1})$.

The following definition is provided for later analysis.

Definition 7.1 *[**Non-weighted** $\mathcal{L}_2$ **gain**] [148] The switched system* (7.2) *is said to have a non-weighted $\mathcal{L}_2$ gain $\overline{\gamma} > 0$, if under zero initial conditions, the following inequality holds:*

$$\int_0^{\infty} y^T(t)y(t)dt \leq \int_0^{\infty} \overline{\gamma}^2 w^T(t)w(t)dt \tag{7.3}$$

for all $t \geq 0$, and all $w(t) \in \mathcal{L}_2^r$.

The following lemma will be used to analyze the $\mathcal{L}_2$ gain.

Lemma 7.1 *[36] All admissible switching laws with dwell time τ_{d} satisfy the following inequality:*

$$N(t_{\mathrm{s}},\, t_{\mathrm{f}}) \leq 1 + \frac{t_{\mathrm{s}} - t_{\mathrm{f}}}{\tau_{\mathrm{d}}}, \quad \forall\, t_{\mathrm{s}} \geq t_{\mathrm{f}} \tag{7.4}$$

where $N(t_{\mathrm{s}},\, t_{\mathrm{f}})$ denotes the number of switchings over the interval $[t_{\mathrm{s}},\, t_{\mathrm{f}})$.

Define the maximum switching delay $\Delta\tau := \max_{i \in \mathbb{N}} \Delta\tau_i$, which is assumed to be known. The set of admissible switching laws with dwell time is denoted by $\mathcal{D}(\tau_{\mathrm{d}})$. Then, the problem to be solved in this work is formulated as follows:

Problem 7.1 *Design an admissible switching law with dwell time such that:*

(i) the system (7.2) *with the knowledge of* $\Delta\tau$ *is globally asymptotically stable for* $w(t) \equiv 0$*;*

(ii) the system (7.2) *with the knowledge of* $\Delta\tau$ *has a non-weighted* $\mathcal{L}_2$ *gain.*

Furthermore, design a switching law with dwell time and a family of mode-dependent state-feedback controllers such that

(iii) the closed-loop system has a non-weighted $\mathcal{L}_2$ *gain.*

7.3 Stability conditions with dwell time constraint

In this section, a novel Lyapunov function is introduced to study the asymptotic stability of (7.2) with $w(t) \equiv 0$. In addition, the LMIs derived from the resulting Lyapunov stability criterion are provided.

7.3.1 A new Lyapunov function

The Lyapunov function most widely used [36, 37, 94] to study the stability of switched linear systems has the form

$$V(t) = x^T(t)P_{\sigma(t)}x(t), \quad V(t_i) \le \mu V(t_i^-), \quad \mu \ge 1 \tag{7.5}$$

for $t \in [t_i,\ t_{i+1})$, where $V(t_i^-)$ represents the left-limit of $V(t)$ at $t = t_i$. This function is continuous between two consecutive switching instants and possibly discontinuous at switching instants. For switched linear systems with switching delays, a new version of (7.5) has been developed [143, 144, 152] as follows:

$$\begin{aligned} &V(t) = x^T(t)P_{\sigma(t)}x(t), \quad V(t_i) \le \mu V(t_i^-), \quad \mu \ge 1 \\ &\dot{V}(t) \le \begin{cases} -\lambda_1 V(t),\ \lambda_1 > 0,\ \forall t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1}) \\ \lambda_2 V(t), \quad \lambda_2 > 0,\ \ \forall t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1}) \end{cases} \end{aligned} \tag{7.6}$$

which has the following property that is different with respect to (7.5): (7.6) might increase during the unmatched intervals and it decreases during the matched intervals, as illustrated in Fig. 7.1. However, the following asymmetry can be noted in state-of-the-art results for stability of switched linear systems with switching delays via (7.6) (c.f. Theorem 1 in [143]): During $t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1})$ the Lyapunov function corresponding to a different subsystem rather than $x^T(t)P_{\sigma(t_i)}x(t)$ should be used, i.e., $x^T(t)P_{\sigma(t_i^-)}x(t)$. In view of this, to reflect the key feature behind unmatched and matched intervals, a new Lyapunov function is proposed for switched linear systems with switching delays:

$$V(t) = \begin{cases} x^T(t)P_{\sigma(t_i^-)}(t)x(t), & \forall t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1}) \\ x^T(t)P_{\sigma(t_i)}(t)x(t), & \forall t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1}) \end{cases} \tag{7.7}$$

which is *continuous at the switching instants* and *discontinuous at the instants when the modes are matched*, as illustrated in Fig. 7.2.

Remark 7.1 The main difference between switched linear systems without switching delays and switched linear systems with switching delays consists in the switching delay between activation of system modes and activation of candidate controller modes. This gives rise to the key feature of switched linear systems with switching delays: the same controller is connected to the previous system mode during the matched interval and to the current system mode during the unmatched interval. In view of this key feature, to solve the stabilization problem using a Lyapunov method, the same positive definite matrix should be adopted in these two intervals to construct the Lyapunov function. This implies that the Lyapunov function is continuous at the switching instants and discontinuous at the instants when the modes are matched, as shown in Fig. 7.2. Note that when the switching delay $\Delta\tau$ is zero, the proposed Lyapunov function (7.7) reduces to the classic Lyapunov function (7.5). In view of this, the Lyapunov function (7.5) can be regarded as a special case of (7.7). □

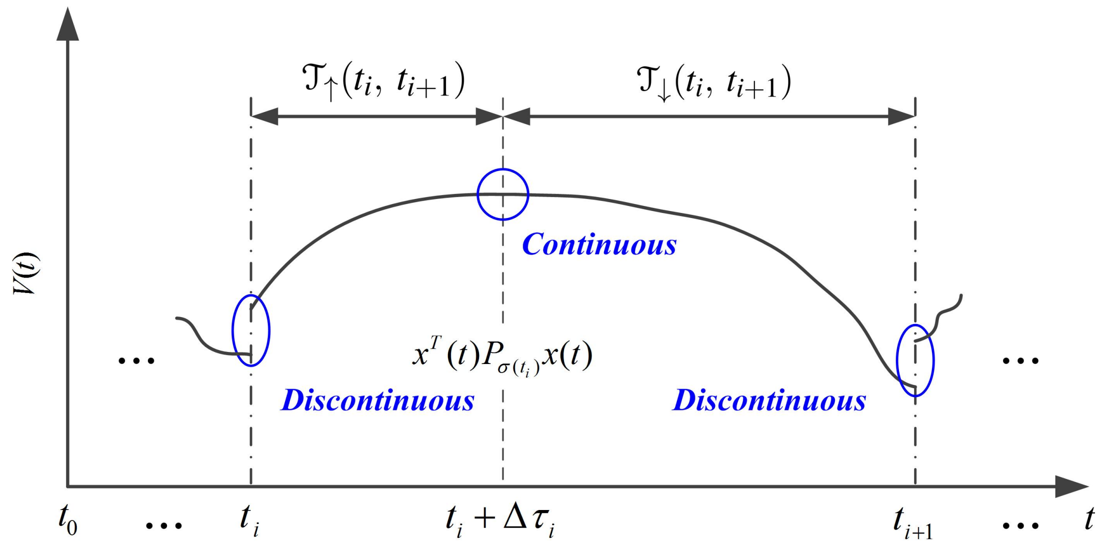

Figure 7.1: The Lyapunov function (7.6).

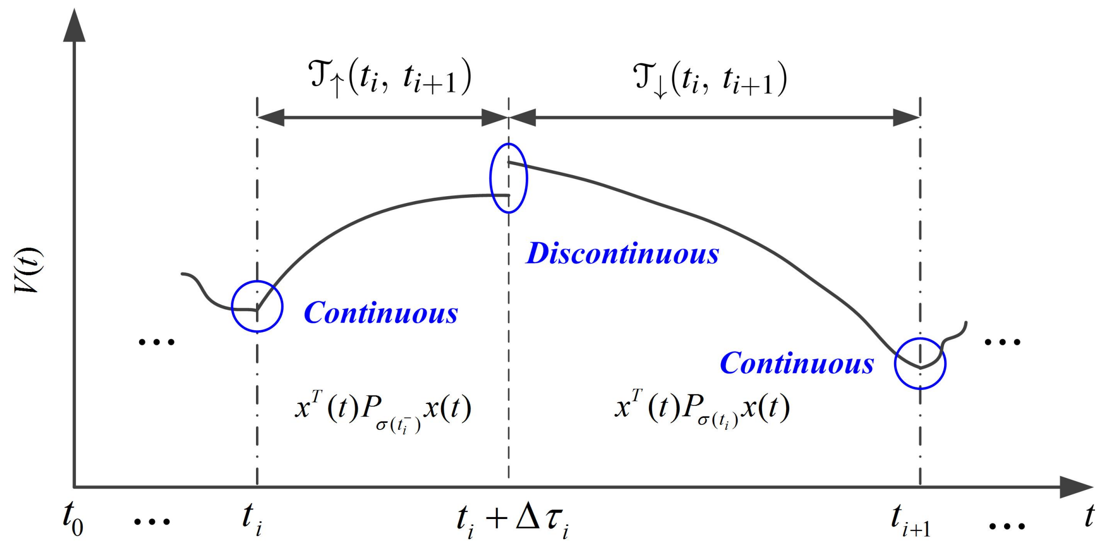

Figure 7.2: The proposed Lyapunov function (7.7).

7.3.2 LMI conditions

Moreover, a time-varying Lyapunov function based on (7.7) can now be constructed by revising the so-called interpolation approach in [1, 123, 126] and by extending it to switched linear systems with switching delays. This gives rise to the new stability criteria as presented below.

7.3.3 Stability results

Theorem 7.2 *Let* $\overline{\lambda}$ *and* $\{\lambda_l\}_{l=0}^{L}$ *be given positive real numbers, where L is a given integer. Suppose there exists a family of positive definite matrices* $P_{p,l}$, $p \in \mathcal{M}$, $l = 0,\ldots,L$, *and a number* $h > 0$ *such that*

$$\overline{A}_{p,q}^{T} P_{q,L} + P_{q,L}\overline{A}_{p,q} - \overline{\lambda} P_{q,L} < 0 \tag{7.8a}$$

$$\Delta P_{l+1,l}^{p}/h + \overline{A}_{p}^{T} P_{p,\ell} + P_{p,\ell}\overline{A}_{p} + \lambda_{p,\ell} P_{p,\ell} < 0 \tag{7.8b}$$

$$\lambda_{p,l} P_{p,l} - \lambda_{l+1} P_{p,l+1} \geq 0 \tag{7.8c}$$

$$\Delta P_{l+1,l}^{p} > 0 \tag{7.8d}$$

$$\overline{A}_{p}^{T} P_{p,L} + P_{p,L}\overline{A}_{p} < 0 \tag{7.8e}$$

$$P_{q,L} - P_{p,0} \geq 0 \tag{7.8f}$$

$$\overline{\lambda}\Delta\tau - \sum_{l=1}^{L} \lambda_l h \leq 0 \tag{7.8g}$$

with $\Delta P_{l+1,l}^{p} := P_{p,l+1} - P_{p,l}$ *for* $\ell = l,\ l+1;\ l = 0,\ldots,L-1;\ \forall q,p \in \mathcal{M}$ *with* $p \neq q$. *Then, the system* (7.2) *with* $w(t) \equiv 0$ *is globally asymptotically stable for any switching signal* $\sigma(\cdot) \in \mathcal{D}(\tau_{\mathrm{d}})$ *with*

$$\tau_{\mathrm{d}} > Lh + \Delta\tau.$$

Proof: To prove Theorem 7.2, the stability condition for switched systems introduced in Theorem 2.2 of Chapter 2 is used. Below we will show in three steps that this stability condition can be guaranteed by the LMIs in (7.8a)–(7.8g):

(a) we construct a quadratic Lyapunov function $V(t)$ in the fashion of (7.7) by interpolating a discrete set of positive definite matrices (obtained from (7.8a)–(7.8g));

(b) within a switching interval $[t_i,\ t_{i+1})$, we show that the increase of the Lyapunov function during unmatched intervals is compensated by the decrease during matched intervals, i.e., $V(t_i) \geq V(t_{i+1}^{-})$;

(c) we exploit continuity of the Lyapunov function at switching instants, i.e., $V(t_{i+1}^{-}) = V(t_{i+1})$.

(a) Without loss of generality, we assume that subsystem p is active for $t \in [t_i,\ t_{i+1})$, $i \in \mathbb{N}$, and subsystem q is active for $t \in [t_{i-1},\ t_i)$. Let us define a time sequence $\{t_{i,l}\}_{l=0}^{L}$, where $t_{i,l+1} - t_{i,l} = h$, $l = 0,\ldots,\ L-1$, $t_{i,0} = t_i + \Delta\tau$, and $t_{i,L} - t_i = \tau_{\mathrm{d}}$, as shown in Fig. 7.3.

To study the properties of the Lyapunov function in (7.7), we partition the interval $[t_i,\ t_{i+1})$ into three subintervals: $[t_i,\ t_{i,0})$, $[t_{i,0},\ t_{i,L})$, and $[t_{i,L},\ t_{i+1})$. Using linear interpolation, we con-

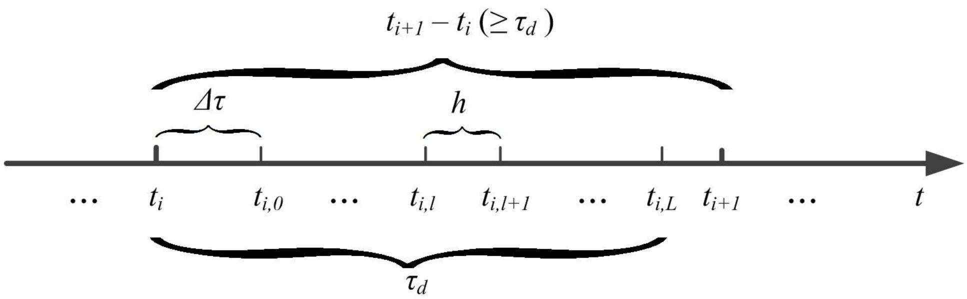

Figure 7.3: The time sequence between two consecutive switching instants.

struct the following time-varying positive definite matrix $P_p(t)$, for $t \in [t_{i,0},\ t_{i+1})$

$$P_p(t) = \begin{cases} P_{p,l} + \rho_{i,l}(t)\Delta P^p_{l+1,l}, & t \in [t_{i,l},\ t_{i,l+1}) \\ P_{p,L}, & t \in [t_{i,L},\ t_{i+1}) \end{cases}$$

where $\rho_{i,l}(t) = (t - t_{i,l})/h$ with $l = 0, \ldots, L-1$. Then, the Lyapunov function (7.7) becomes, for $t \in [t_i,\ t_{i+1})$

$$V(t) = \begin{cases} x^T(t)P_{q,L}x(t), & t \in [t_i,\ t_{i,0}) \\ x^T(t)P_p(t)x(t), & t \in [t_{i,0},\ t_{i+1}) \end{cases} \tag{7.9}$$

which is continuous at switching instants, and discontinuous at the instant $t_{i,0}$ when the controller and subsystem are matched.

(b) First, we consider the subinterval $[t_i,\ t_{i,0})$. According to (7.8a), the derivative of $V(t)$ in (7.9) is

$$\dot{V}(t) = x^T(t)(\overline{A}^T_{p,q}P_{q,L} + P_{q,L}\overline{A}_{p,q})x(t) < \overline{\lambda}V_p(t).$$

At the instant $t_{i,0}$, it follows from (7.8f) that

$$V(t^-_{i,0}) - V_p(t_{i,0}) = x^T(t)(P_{q,L} - P_{p,0})x(t) \geq 0$$

which implies that the Lyapunov function is non-increasing at time instants of discontinuity. Next, for the second subinterval $[t_{i,0},\ t_{i,L})$, according to (7.8b)–(7.8c), we have $\dot{V}(t) = x^T(t)\mathscr{P}(t)x(t)$, for $t \in [t_{i,l},\ t_{i,l+1})$, where

$$\begin{aligned} \mathscr{P}(t) &= \overline{A}^T_p P(t) + P(t)\overline{A}_p + \Delta P^p_{l+1,l}/h \\ &= \eta_1\left(\Delta P^p_{l+1,l}/h + P_{p,l}\overline{A}_p + \overline{A}^T_p P_{p,l}\right) \\ &\quad + \eta_2\left(\Delta P^p_{l+1,l}/h + P_{p,l+1}\overline{A}_p + \overline{A}^T_p P_{p,l+1}\right) \\ &< -\left(\eta_1\lambda_l P_{p,l} + \eta_2\lambda_{l+1}P_{p,l+1}\right) \\ &< -\lambda_{l+1}P_{p,l+1} - \eta_1\left(\lambda_l P_{p,l} - \lambda_{l+1}P_{p,l+1}\right) \\ &< -\lambda_{l+1}P_{p,l+1} \end{aligned} \tag{7.10}$$

with $\eta_1 = 1-(t-t_{p,l})/h$, $\eta_2 = 1-\eta_1$. Moreover, the inequality (7.8d) implies that $P_{p,l+1} - P(t) \geq 0$, which combined with (7.10) leads to

$$\dot{V}(t) < -\lambda_{l+1}x^T(t)P(t)x(t), \quad t \in [t_{i,l},\ t_{i,l+1})$$

for $l = 0, \ldots, L-1$. This implies that (7.9) is decreasing exponentially with different rates $\{\lambda_l\}_{l=1}^{L}$ during different intervals $[t_{i,l},\ t_{i,l+1})$. Then, we have

$$V_p(t_{i,L}) < \exp(\sum_{l=0}^{L} \lambda_l h) V_p(t_{i,0}).$$

Considering the properties of (7.9) during the first subinterval $[t_i,\ t_{i,0})$ and at the time instant $t_{i,0}$ of discontinuity, we obtain

$$V(t_i) < \exp(-\overline{\lambda}\Delta\tau + \sum_{l=0}^{L} \lambda_l h) V_p(t_{i,L})$$

which implies that $V(t_{i,L}) \leq V(t_i)$ by (7.8g). Finally, we consider the third subinterval $t \in [t_{i,L},\ t_{i+1})$. Since the matrix $P_p(t)$ is constant during the third subinterval, it holds that

$$\dot{V} = x^T (A_p^T P_{p,L} + P_{p,L} A_p) x < 0$$

according to (7.8e), i.e., $V(t_{i+1}^-) < V(t_{i,L})$. Then, it means that the increase of the Lyapunov function (7.9) during the unmatched interval is compensated by the decrease during the matched interval, i.e., $V(t_{i+1}^-) < V(t_i)$.

(c) Since the Lyapunov function (7.9) is continuous at the switching instants, i.e., $V(t_{i+1}^-) = V(t_{i+1})$, we have $V(t_{i+1}) \leq V(t_i)$, which satisfies that stability condition by Branicky [9]. This completes the proof. □

The LMIs (7.8a)–(7.8g) might be difficult to solve due to the large number of design parameters λ_l, $l = 0, \ldots, L$ when a large integer L is chosen. Therefore, a more convenient option is to use a common rate of decrease during matched intervals, i.e., $\lambda_0 = \cdots = \lambda_L = \beta$. This simplification gives rise to the following corollary, which involves only two design parameters and in return may give conservative results as compared with *Theorem* 7.2.

Corollary 7.1 *Let α and β be given positive constants. Suppose there exists a family of positive definite matrices $P_{p,l}$, $p \in \mathcal{M}$, $l = 0, \ldots, L$ with a given integer L, and a number $h > 0$, such that*

$$\overline{A}_{p,q}^T P_{q,L} + P_{q,L} \overline{A}_{p,q} - \alpha P_{q,L} < 0 \tag{7.11a}$$
$$\Delta P_{l+1,l}^p / h + \overline{A}_p^T P_{p,\ell} + P_{p,\ell} \overline{A}_p + \beta P_{p,\ell} < 0 \tag{7.11b}$$
$$\Delta P_{l+1,l}^p > 0 \tag{7.11c}$$
$$\overline{A}_p^T P_{p,L} + P_{p,L} \overline{A}_p + \beta P_{p,L} < 0 \tag{7.11d}$$
$$P_{q,L} - P_{p,0} \geq 0 \tag{7.11e}$$

for $\ell = l,\ l+1$; $l = 0, \ldots, L-1$; $\forall q, p \in \mathcal{M}$ with $p \neq q$. Then, the system (7.2) *with* $w(t) \equiv 0$ *is globally asymptotically stable for any switching law $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$ with*

$$\tau_\mathrm{d} > \max\left(\Delta\tau + hL, (\alpha+\beta)\Delta\tau/\beta\right).$$

Proof: The proof has the following three steps in a similar vein to the one for *Theorem* 7.2.

(a) Without loss of generality, we assume that during the switching interval $[t_i, t_{i+1})$, $i \in \mathbb{N}$, subsystem p is active, and during the switching interval $[t_{i-1}, t_i)$, $i \in \mathbb{N}_+$, subsystem q is active. In order to enforce that the increase of the Lyapunov function over the unmatched interval is compensated by a decrease in the matched interval, we define a new positive number $\hat{h}$ as

$$\hat{h} = \begin{cases} \alpha\Delta\tau/(\beta L), & \text{if } \beta L h < \alpha\Delta\tau \\ h, & \text{otherwise.} \end{cases}$$

It is evident that $\hat{h} \geq h$, which implies that

$$\frac{\Delta P^p_{l+1,l}}{h} - \frac{\Delta P^p_{l+1,l}}{\hat{h}} > 0.$$

Considering that

$$\frac{\Delta P^p_{l+1,l}}{h} > 0$$

due to (7.11c), it can be shown that if (7.11b) holds, then

$$\frac{\Delta P^p_{l+1,l}}{\hat{h}} + \overline{A}_p^T P_{p,\ell} + P_{p,\ell}\overline{A}_p + \beta P_{p,\ell} < 0.$$

Let us define a time sequence $\{t_{i,l}\}_{l=0}^{L}$, where $t_{i,l+1} - t_{i,l} = \hat{h}$, $l = 0, \ldots, L-1$, $t_{i,0} = t_i + \Delta\tau$, and $t_{i,L} - t_i = \tau_{\mathrm{d}}$, as shown in Fig. 7.4.

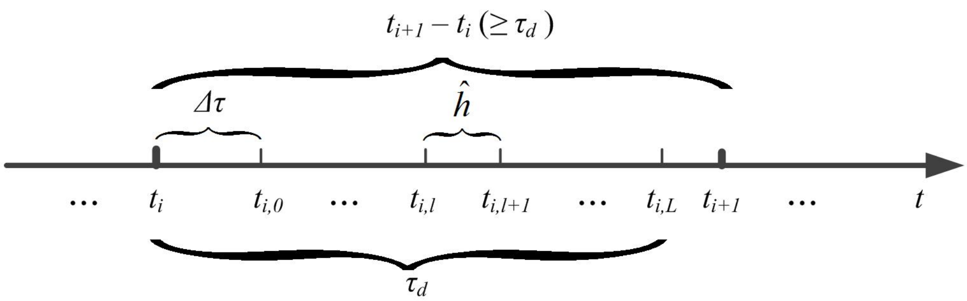

Figure 7.4: The time sequence between two consecutive switching instants.

Similarly, we partition the interval $[t_i, t_{i+1})$ into three subintervals: $[t_i, t_{i,0})$, $[t_{i,0}, t_{i,L})$, and $[t_{i,L}, t_{i+1})$. The time-varying positive definite matrix $P_p(t)$ is

$$P_p(t) = \begin{cases} P_{p,l} + \hat{\rho}_{i,l}(t)\Delta P^p_{l+1,l}, & t \in [t_{i,l}, t_{i,l+1}) \\ P_{p,L}, & t \in [t_{i,L}, t_{i+1}) \end{cases} \tag{7.12}$$

where $\hat{\rho}_{i,l}(t) = (t - t_{i,l})/\hat{h}$. Then, we construct a Lyapunov function similar with (7.9) using (7.12).

(b) For the first subinterval $t \in [t_i, t_{i,0})$, the derivative of the Lyapunov function is

$$\dot{V}(t) \leq \alpha V(t), \quad t \in [t_i, t_{i,0})$$

according to to (7.11a), and for $t \in [t_{i,0},\ t_{i+1})$, according to (7.11b)–(7.11c), it holds that

$$\dot{V}(t) \leq -\beta V(t), \quad t \in [t_{i,0},\ t_{i+1})$$

using similar steps as (7.10) in the proof of *Theorem* 7.2. Since the Lyapunov function is non-increasing at the instant $t_{i,0}$, using the dwell time $\tau_\mathrm{d} > \max\big(\Delta\tau + hL, (\alpha+\beta)\Delta\tau/\beta\big)$, we can guarantee that $V(t_{i+1}^-) \leq V(t_i)$.

(c) Finally, we refer to the same reasoning as the third part in *Theorem* 7.2. This completes the proof. □

Remark 7.3 As noted in [143], for a stable switched system with switching delays, one can always find α (characterizing the rate of the exponential increase) big enough and β (characterizing the rate of the exponential decrease) small enough to certify stability; a similar situation arises also in our case. In addition, according to the results in [1, 126, 145], given β satisfying (7.11), there exists a lower bound $\underline{h}$ for h such that feasibility occurs for any $h \geq \underline{h}$. This suggests the use of a sequence of a line search approach to solve (7.11), where the scalars α, β, h are searched according to the aforementioned guidelines and (7.11) reduces to an LMI for fixed α, β, h. □

7.4 $\mathcal{L}_2$ analysis and synthesis

In this section, a non-weighted $\mathcal{L}_2$ gain for switched linear systems with switching delays is derived from the Lyapunov function (7.7). Moreover, the LMIs for controller design are provided based on *Corollary* 7.1.

7.4.1 Non-weighted $\mathcal{L}_2$ gain

Lemma 7.2 *Let α and β be given positive constants. Suppose there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, and two class-$\mathcal{K}_\infty$ functions κ_1 and κ_2 such that, for $t \in [t_i,\ t_{i+1})$, $\forall i \in \mathbb{N}$, we have*

$$\begin{aligned}
&V((t_i + \Delta\tau_i)^-) \geq V(t_i + \Delta\tau_i)\\
&V(t_i^-) = V(t_i)\\
&\kappa_1(|x(t)|) \leq V(x(t)) \leq \kappa_2(|x(t)|)
\end{aligned}$$

$\forall t \geq 0$, *and*

$$\dot{V}(t) \leq \begin{cases} \alpha V(t) - \Gamma(t), & t \in \mathcal{T}_\uparrow(t_i,\ t_{i+1}) \\ -\beta V(t) - \Gamma(t), & t \in \mathcal{T}_\downarrow(t_i,\ t_{i+1}) \end{cases} \tag{7.13}$$

where $\Gamma(t) = y^T(t)y(t) - \gamma^2 w^T(t)w(t)$. *Then, the system* (7.2) *achieves a non-weighted $\mathcal{L}_2$ gain*

$$\overline{\gamma} = \sqrt{\frac{\beta\tau_\mathrm{d} e^{(\alpha+\beta)\Delta\tau}}{\beta\tau_\mathrm{d} - (\alpha+\beta)\Delta\tau}}\gamma \tag{7.14}$$

for any switching signal $\sigma(\cdot) \in \mathcal{D}(\tau_\mathrm{d})$ *with*

$$\tau_\mathrm{d} > \max\big(\Delta\tau + hL, (\alpha+\beta)\Delta\tau/\beta\big).$$

Proof: Consider an interval $[t_i,\ t_{i+1})$, $i \in \mathbb{N}$. We represent the total unmatched interval and matched interval between $[t_\mathrm{s},\ t_\mathrm{f})$ by $\mathscr{T}^{\uparrow}(t_\mathrm{s},\ t_\mathrm{f})$ and $\mathscr{T}^{\downarrow}(t_\mathrm{s},\ t_\mathrm{f})$, respectively. To keep the mathematical derivation concise, let us use the following notation

$$\mathbf{E}(a,b) := e^{\alpha \mathscr{T}^{\uparrow}(a,\ b) - \beta \mathscr{T}^{\downarrow}(a,\ b)}$$

with $a > b \geq 0$. Since

$$V(t_i + \Delta\tau_i) - V((t_i + \Delta\tau_i)^-) \leq 0$$

for any $i \in \mathbb{N}$, it follows from (7.13) that

$$\begin{aligned}
V(t) &\leq V(t_i)\mathbf{E}(t_i, t) - \int_{t_i}^{t} \mathbf{E}(s,t)\Gamma(s)ds \\
&\leq \left(V(t_{i-1})\mathbf{E}(t_i, t) - \int_{t_{i-1}}^{t_i} \mathbf{E}(s,t_i)\Gamma(s)ds\right)\mathbf{E}(t_i, t) - \int_{t_i}^{t} \mathbf{E}(s,t)\Gamma(s)ds \\
&= V(t_{i-1})\mathbf{E}(t_{i-1}, t) - \int_{t_{i-1}}^{t} \mathbf{E}(s,t)\Gamma(s)ds \\
&\ \ \vdots \\
&\leq V(t_0)\mathbf{E}(t_0, t) - \int_{t_0}^{t} \mathbf{E}(s,t)\Gamma(s)ds.
\end{aligned} \tag{7.15}$$

Considering the initial condition $V(t_0) = 0$, and $V(t) \geq 0$, and substituting

$$\Gamma(t) = y^T(t)y(t) - \gamma^2 w^T(t)w(t)$$

into (7.15) gives

$$\int_{t_0}^{t} \mathbf{E}(s,t)y^T(s)y(s)ds \leq \int_{t_0}^{t} \mathbf{E}(s,t)\gamma^2 w^T(s)w(s)ds$$

where the left-hand side is given by

$$\begin{aligned}
&\int_{t_0}^{t} \mathbf{E}(s,t)y^T(s)y(s)ds \\
&= \int_{t_0}^{t} e^{(\alpha+\beta)\mathscr{T}^{\uparrow}(s,\ t) - \beta(t-s)} y^T(s)y(s)ds \\
&\geq \int_{t_0}^{t} e^{-\beta(t-s)} y^T(s)y(s)ds
\end{aligned} \tag{7.16}$$

and the right-hand side is

$$\begin{aligned}
&\int_{t_0}^{t} \mathbf{E}(s,t)\gamma^2 w^T(s)w(s)ds \\
&= \int_{t_0}^{t} e^{(\alpha+\beta)\mathscr{T}^{\uparrow}(s,\ t) - \beta(t-s)} \gamma^2 w^T(s)w(s)ds \\
&\leq \int_{t_0}^{t} e^{N(s,\ t)(\alpha+\beta)\Delta\tau - \beta(t-s)} \gamma^2 w^T(s)w(s)ds \\
&\leq \int_{t_0}^{t} e^{(1+\frac{t-s}{\tau_\mathrm{d}})(\alpha+\beta)\Delta\tau - \beta(t-s)} \gamma^2 w^T(s)w(s)ds
\end{aligned} \tag{7.17}$$

$$\leq \int_{t_0}^{t} e^{(\alpha+\beta)\Delta\tau} e^{\frac{(\alpha+\beta)\Delta\tau-\beta\tau_{\mathrm{d}}}{\tau_p}(t-s)} \gamma^2 w^T(s) w(s) ds$$

where the second inequality in (7.17) holds due to (7.4). Let $t_0 = 0$ according to the definition of the non-weighted $\mathcal{L}_2$ gain (Definition 7.1). Integrating (7.16) and (7.17) for t going from 0 to ∞, we have

$$\begin{aligned} &\int_0^{\infty}\int_0^{t} e^{-\beta(t-s)} y^T(s) y(s) ds dt \\ &= \int_0^{\infty} y^T(s) y(s) \left(\int_s^{\infty} e^{-\beta(t-s)} dt\right) ds \\ &= \frac{1}{\beta}\int_0^{\infty} y^T(s) y(s) ds \end{aligned} \tag{7.18}$$

and

$$\begin{aligned} &\int_0^{\infty}\int_0^{t} e^{(\alpha+\beta)\Delta\tau} e^{-\frac{\Delta T}{\tau_{\mathrm{d}}}(t-s)} \gamma^2 w^T(s) w(s) ds\, dt \\ &= e^{(\alpha+\beta)\Delta\tau}\int_0^{\infty}\left(\int_s^{\infty} e^{-\frac{\Delta T}{\tau_{\mathrm{d}}}(t-s)} dt\right)\gamma^2 w^T(s) w(s) ds \\ &= \frac{\tau_{\mathrm{d}}}{-\Delta T} e^{(\alpha+\beta)\Delta\tau}\gamma^2 \int_0^{\infty} w^T(s) w(s) ds \end{aligned} \tag{7.19}$$

due to $\Delta T := (\alpha+\beta)\Delta\tau - \beta\tau_{\mathrm{d}} < 0$.

Combining (7.18) and (7.19) leads to

$$\frac{1}{\beta}\int_0^{\infty} y^T(s) y(s) ds \leq \frac{\tau_{\mathrm{d}} e^{(\alpha+\beta)\Delta\tau}}{-\Delta T}\gamma^2 \int_0^{\infty} w^T(s) w(s) ds$$

which indicates that a non-weighted $\mathcal{L}_2$ gain as (7.14) for the system (7.2) is guaranteed. This completes the proof. □

Theorem 7.4 *Let α and β be given positive constants. Suppose there exists a family of positive definite matrices $P_{p,l}$, $p \in \mathcal{M}$, $l = 0, \ldots, L$, such that*

$$\begin{gathered} \begin{bmatrix} \Phi_{p,q} & P_{q,L}E_p & \overline{C}_{p,q}^T \\ * & -\gamma^2 I & F_p^T \\ * & * & -I \end{bmatrix} < 0, \quad \begin{bmatrix} \Theta_p & P_{p,\ell}E_p & \overline{C}_p^T \\ * & -\gamma^2 I & F_p^T \\ * & * & -I \end{bmatrix} < 0 \\ \begin{bmatrix} \Psi_p & P_{p,L}E_p & \overline{C}_p^T \\ * & -\gamma^2 I & F_p^T \\ * & * & -I \end{bmatrix} < 0, \quad \begin{array}{l} \Delta P_{l+1,l}^p > 0 \\ P_{q,L} - P_{p,0} \geq 0 \end{array} \end{gathered} \tag{7.20}$$

for $\ell = l,\ l+1$; $l = 1, \ldots, L-1$; $\forall p, q \in \mathcal{M}$ with $p \neq q$, where

$$\begin{aligned} \Phi_{p,q} &= \overline{A}_{p,q}^T P_{q,L} + P_{q,L}\overline{A}_{p,q} - \alpha P_{q,L} \\ \Theta_p &= \Delta P_{l+1,l}^p / \hat{h} + P_{p,\ell}\overline{A}_p + \overline{A}_p^T P_{p,\ell} + \beta P_{p,\ell} \\ \Psi_p &= P_{p,L}\overline{A}_p + \overline{A}_p^T P_{p,L} + \beta P_{p,L}. \end{aligned}$$

Then, the switched linear system with switching delays (7.2) *achieves a non-weighted $\mathcal{L}_2$ gain*

$\overline{\gamma}$ (7.14) *for any switching law* $\sigma(\cdot) \in \mathscr{D}(\tau_{\mathrm{d}})$ *with*

$$\tau_{\mathrm{d}} > \max\big(\Delta\tau + hL, (\alpha+\beta)\Delta\tau/\beta\big).$$

Proof: According to the standard derivation of the bounded real lemma for linear systems [8] and the definition of $P_p(t)$ in (7.12), it can be verified that (7.20) leads to the following:

$$\begin{aligned} \dot{V}(t) &\leq \alpha V(t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t), \qquad t \in \mathscr{T}_{\uparrow}(t_i,\ t_{i+1}) \\ \dot{V}(t) &\leq -\beta V(t) + y^T(t)y(t) - \gamma^2 w^T(t)w(t), \quad t \in \mathscr{T}_{\downarrow}(t_i,\ t_{i+1}) \end{aligned}$$

which is in the same form as (7.13). Furthermore, $V(t)$ is continuous at the switching instants, and non-increasing at the instants when the modes are matched. This means that *Lemma* 7.2 holds, and we can guarantee a non-weighted $\mathcal{L}_2$ gain for switched systems (7.2) via the dwell time $\tau_{\mathrm{d}} > \max\big(\Delta\tau + hL, (\alpha+\beta)\Delta\tau/\beta\big)$. □

7.4.2 Robust H_∞ control design

Theorem 7.5 *Let α and β be given positive constants. Suppose there exists a family of positive definite matrices $Q_{p,l}$, a family of vectors $U_{p,l}$, $p \in \mathcal{M}$, $l = 0,\ldots,L$, and a positive number h such that*

$$\begin{gathered} \begin{bmatrix} \mathscr{H}_{p,q} & E_p & \mathscr{Z}_{p,q} \\ * & -\gamma^2 I & F_p^T \\ * & * & -I \end{bmatrix} < 0, \quad \begin{bmatrix} \Xi_p & E_p & \Lambda_p \\ * & -\gamma^2 I & F_p^T \\ * & * & -I \end{bmatrix} < 0 \\ \begin{bmatrix} \Omega_p & E_p & \Upsilon_p \\ * & -\gamma^2 I & F_p^T \\ * & * & -I \end{bmatrix} < 0, \quad \begin{array}{l} \Delta Q^p_{l+1,l} < 0 \\ -Q_{q,L} + Q_{p,0} \geq 0 \end{array} \end{gathered} \tag{7.21}$$

for $\ell = l,\ l+1; l = 0,\ldots,L-1; \forall p,q \in \mathcal{M}$ with $p \neq q$, where

$$\begin{aligned} \mathscr{H}_{p,q} &= Q_{q,L}A_p^T + A_pQ_{q,L} + U_{q,L}^TB_p^T + B_pU_{q,L} - \alpha Q_{q,L} \\ \mathscr{Z}_{p,q} &= Q_{q,L}C_p^T + U_{q,L}^TD_p^T \\ \Xi_p &= \Delta Q^p_{l+1,l}/\hat{h} + Q_{p,\ell}A_p^T + A_pQ_{p,\ell} + U_{p,\ell}^TB_p^T \\ &\quad + B_pU_{p,\ell} + \beta P_{p,\ell} \\ \Lambda_p &= Q_{p,\ell}C_p^T + U_{p,\ell}^TD_p^T \\ \Omega_p &= Q_{p,L}A_p^T + A_pQ_{p,L} + U_{p,L}^TB_p^T + B_pU_{p,L} + \beta_pP_{p,L} \\ \Upsilon_p &= Q_{p,L}C_p^T + U_{p,L}^TD_p^T. \end{aligned}$$

Then, there exists a family of mode-dependent state-feedback controllers $u(t) = G_{\sigma(t)}x(t)$ with the maximum switching delay $\Delta\tau$ such that the system (7.2) *achieves a non-weighted $\mathcal{L}_2$ gain* $\overline{\gamma}$ (7.14) *for any switching law* $\sigma(\cdot) \in \mathscr{D}(\tau_{\mathrm{d}})$ *with*

$$\tau_{\mathrm{d}} > \max\big(\Delta\tau + hL, (\alpha+\beta)\Delta\tau/\beta\big).$$

Additionally, the gains of state feedback controllers with switching delay can be obtained as

$$G_p(t) = \begin{cases} \left[U_{p,l} + \hat{\rho}_{i,l}(t)\Delta U^p_{l+1,l}\right] \cdot \\ \quad \left[Q_{p,l} + \hat{\rho}_{i,l}(t)\Delta Q^p_{l+1,l}\right]^{-1}, & t \in [t_{i,l},\, t_{i,l+1}) \\ U_{p,L} Q^{-1}_{p,L}, & t \in [t_{i,L},\, t_{i+1,0}) \end{cases} \tag{7.22}$$

for $l = 0, \ldots, L-1$, *where* $\Delta U^p_{l+1,l} = U_{p,l} - U_{p,l+1}$, $\hat{\rho}_{i,l}(t) = (t - t_{i,l})/\hat{h}$ *with* $t_{i,l}$ *shown in Fig. 7.4.*

Proof: Let $Q_{p,l} = P^{-1}_{p,l}$, for $l = 0, \ldots, L$. Substituting $\overline{A}_{p,q}$, $\overline{A}_p$, $\overline{C}_{p,q}$ and $\overline{C}_p$ in (7.2) into (7.20), and then pre-multiplying and post-multiplying by diag$\{Q_{p,l},\, I,\, I\}$ from both sides, the state feedback gains $G_p(t)$ are obtained. □

Remark 7.6 The following difference must be remarked between the results in [1, 123, 126] and the results of this work. In [1, 123, 126], it has been shown that by increasing L, a less conservative dwell time can be found. In Corollary 7.1, when $\Delta\tau + hL \le (\alpha + \beta)\Delta\tau/\beta$, the dwell time τ_d is not affected by the choice of L. However, increasing L might reduce conservativeness in terms of $\mathcal{L}_2$ gain, as illustrated in the example of *Section 7.5.* □

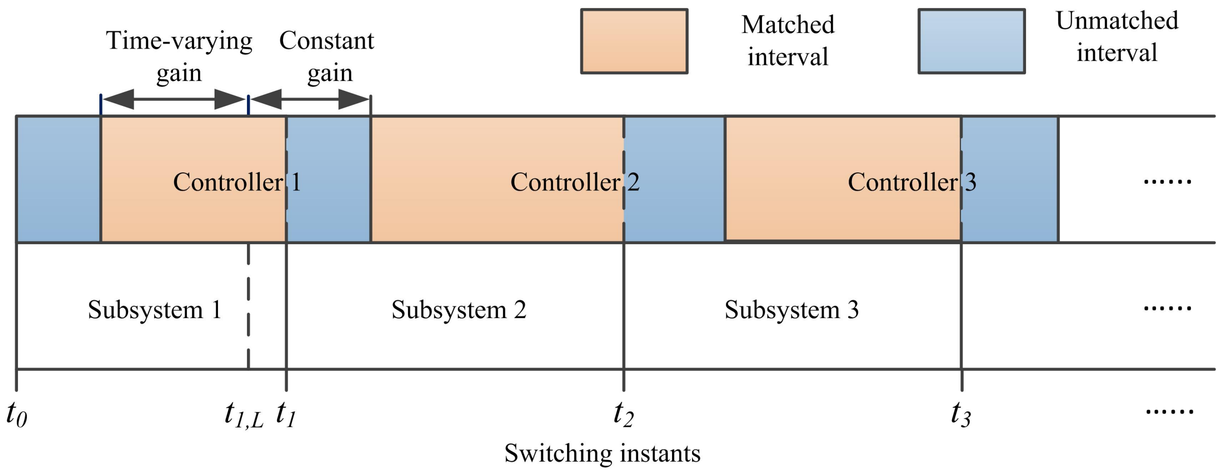

Figure 7.5: The control scheme of (7.2).

Remark 7.7 Derived from the novel Lyapunov function (7.9), the state-feedback gains of the mode-dependent controller are designed to be time-varying only during the matched interval according to (7.22). This implies that after the switching, the controller associated to the subsystem has a constant gain during the unmatched interval, as illustrated in Fig. 7.5. This can simplify the analysis and design of the closed-loop system formed, during an unmatched interval whose actual length is unknown, by the currently active subsystem and the constant-gain controller. □

Remark 7.8 Thanks to the convexity of the proposed stability analysis and H_∞ control design with respect to the system matrices A_p, the stability condition (7.11) and the stabilization condition (7.21) can be extended readily to switched linear system (7.1) with polytopic parametric uncertainties in a similar vein to ones in [1]. □

7.5 Example

In this section, the following switched linear system with maximum switching delay $\Delta\tau = 2$ is adopted to illustrate the proposed results:

$$A_1 = \begin{bmatrix} 0.9 & -5.8 \\ 2.75 & 0.9 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 2 \\ 2.1 & -1.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1.5 \\ 2.2 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 1.85 \\ 1.75 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.45 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.1 \\ 0.5 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 0.2 & 0.6 \end{bmatrix}^T, \quad D_1 = D_2 = 1.5, \quad F_1 = F_2 = 0.65.$$

A. *Non-weighted $\mathcal{L}_2$ gain using* (7.7)

In this subsection, different choices for the value of L are considered to illustrate the results in this chapter.

(a) ($L = 1$) We select $L = 1$, $\alpha = 0.26$, $\beta = 0.2$. After solving the convex optimization problem (7.21), we obtain $\gamma = 1.2746$, $h = 0.06$, $\hat{h} = 2.6$, $\tau_{\mathrm{d}}^* =: (\alpha+\beta)\Delta\tau/\beta = 4.6$, and the following matrices and vectors:

$$Q_{1,0} = \begin{bmatrix} 4.8481 & -0.0466 \\ -0.0466 & 0.8858 \end{bmatrix}, \quad Q_{1,1} = \begin{bmatrix} 4.8420 & -0.0613 \\ -0.0613 & 0.8503 \end{bmatrix}$$
$$Q_{2,0} = \begin{bmatrix} 6.3250 & -0.8206 \\ -0.8206 & 1.2390 \end{bmatrix}, \quad Q_{2,1} = \begin{bmatrix} 4.2490 & -0.1715 \\ -0.1715 & 0.8598 \end{bmatrix}$$
$$U_{1,0} = \begin{bmatrix} -3.6725 & -0.7633 \end{bmatrix}, \quad U_{1,1} = \begin{bmatrix} -3.7385 & -0.6484 \end{bmatrix}$$
$$U_{2,0} = \begin{bmatrix} -1.1715 & -1.0804 \end{bmatrix}, \quad U_{2,1} = \begin{bmatrix} -2.8473 & -0.1361 \end{bmatrix}.$$

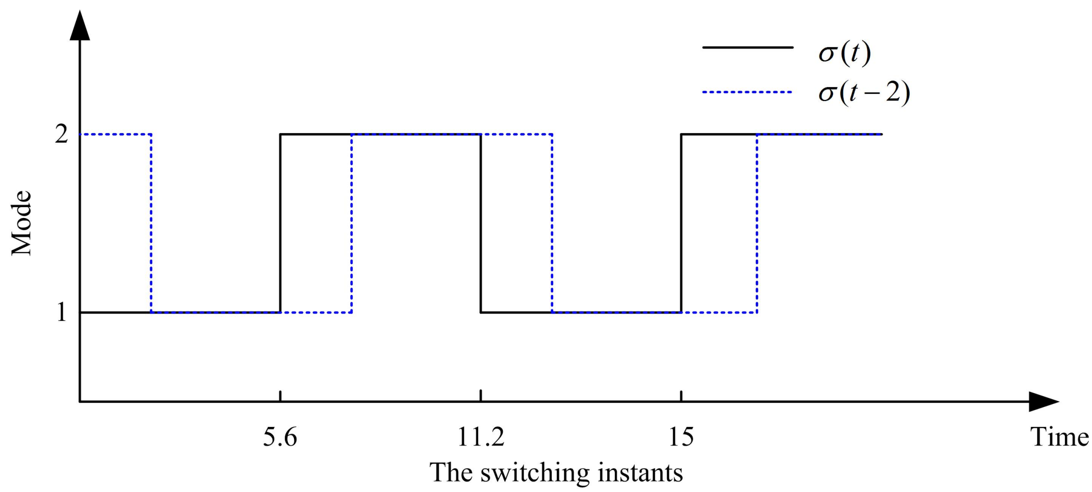

Figure 7.6: The switching signal.

Selecting $t_{i+1} - t_i = \tau_{\mathrm{d}} = 5.6 > \tau_{\mathrm{d}^*}$, $i \in \mathbb{N}$, we have the non-weighted $\mathcal{L}_2$ gain $\overline{\gamma} = 4.7865$ according to (7.14). Then, using (7.22), the controller gains for the two system modes are obtained as follows:

$$G_p(t) = \begin{cases} \left[(t-t_{i,0})\Delta U_p/\hat{h} + U_{p,0}\right] \cdot \\ \quad \left[(t-t_{i,0})\Delta Q_p/\hat{h} + Q_{p,0}\right]^{-1}, & t \in [t_{i,0},\ t_{i,1}) \\ U_{p,1}Q_{p,1}^{-1}, & t \in [t_{i,1},\ t_{i+1,0}) \end{cases} \tag{7.23}$$

for $p \in \{1,2\}$, where $\Delta U_p = U_{p,1} - U_{p,0}$, and $\Delta Q_p = Q_{p,1} - Q_{p,0}$. Let the disturbance $w(t) \equiv 0$, and the initial condition $x_0 = [2\ 1]^T$. We design the switching signal as in 7.6. Then, the resulting Lyapunov function is given in Fig. 7.7, which shows that when the controller mode and the system mode are matched, i.e., at $t = 2$, the Lyapunov function is decreasing, and at the switching instant $t = 5.6$, the Lyapunov function is continuous. In addition, the Lyapunov function tends to zero, as predicted by the global asymptotic stability results.

For the disturbance, let us consider for example $w(t) = 0.5\exp(-0.2t)$, and let the initial condition be $x_0 = [2\ 1]^T$. Adopting the controllers (7.23) with $\mathcal{L}_2$ gain $\overline{\gamma} = 4.7865$ gives rise to the state response shown in Fig. 7.8, which is stable.

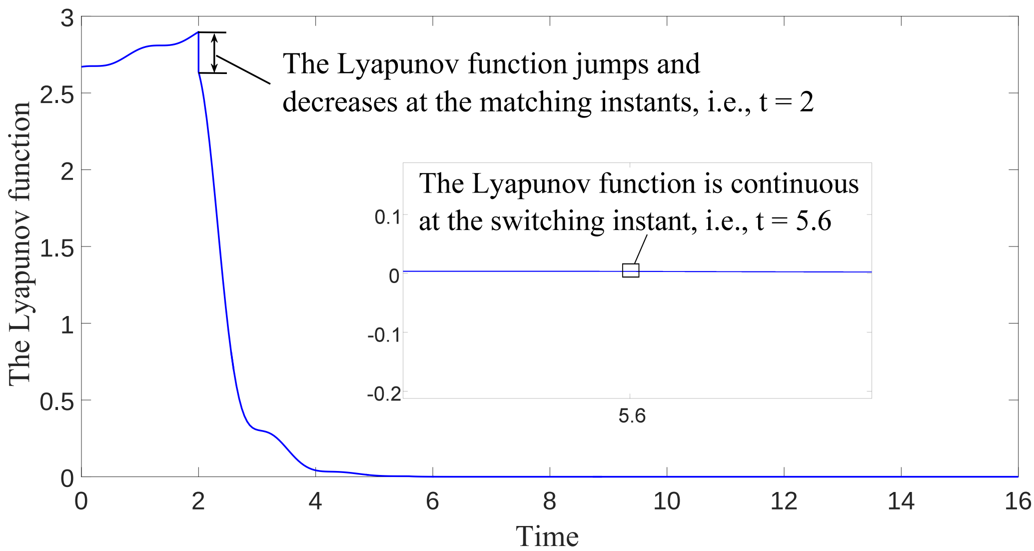

Figure 7.7: The proposed Lyapunov functions with a zoomed detail around $t = 5.6$.

(b) ($L > 1$) Now we choose different values of $L \in \{1, 5, 20, 90, 100\}$, and $\alpha = 0.26$, $\beta = 0.2$. By solving the convex optimization problem (7.21), we get different $\mathcal{L}_2$ gain $\overline{\gamma}$ as shown in Table 7.1. It can be observed that a less conservative $\mathcal{L}_2$ gain is obtained as L increases.

Table 7.1: Non-weighted $\mathcal{L}_2$ gain $\overline{\gamma}$ for different values of L.

L	1	5	20	90	100
γ	1.2769	1.0546	1.0474	1.0436	1.0435
$\overline{\gamma}$	4.7865	3.9116	3.9119	3.9118	3.9118

B. Comparison between (7.6) *and* (7.7)

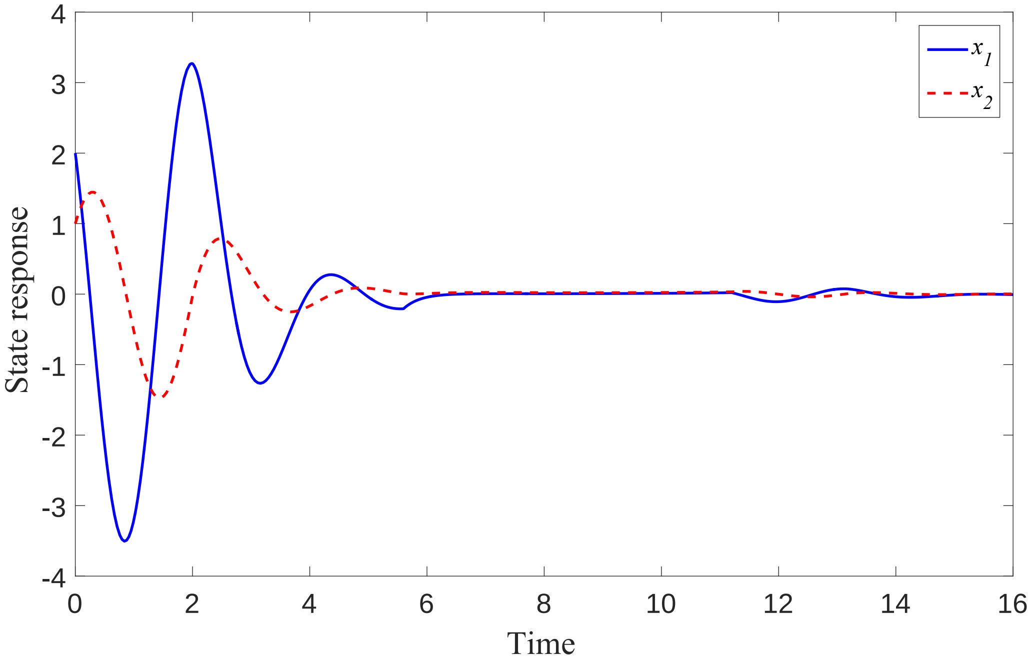

Figure 7.8: State response of closed-loop systems with controllers (7.23).

The key properties about continuity and discontinuity of (7.6) and (7.7) are compared herein. To facilitate understanding of the comparison between (7.6) and (7.7), we let $L = 1$. First, we adopt the same technique to develop time-varying matrices $P_p(t)$ for (7.6) and derive the conditions for designing the two mode-dependent controllers: substituting $Q_{p,0}$ with $Q_{q,L}$ in $\mathscr{H}_{p,q}$, and replacing $-Q_{q,L}+Q_{p,0} \geq 0$ with $-Q_{p,L}+Q_{q,0} \geq 0$ in (7.21), $p \neq q \in \{1,2\}$. Therefore, the resulting controller gains are, for $p = 1, 2$

$$G_p(t) = \begin{cases} \left[(t-t_i)\Delta U_p/\hat{h} + U_{p,0}\right] \cdot & \\ \quad \left[(t-t_i)\Delta Q_p/\hat{h} + Q_{p,0}\right]^{-1}, & t \in [t_i,\, t_{i,1}) \\ U_{p,1}Q_{p,1}^{-1}, & t \in [t_{i,1},\, t_{i+1}) \end{cases}$$

which shows that the mode-dependent controllers u_p are active during the interval $[t_i,\, t_{i+1})$. This implies that the mode-dependent controllers designed via (7.6) fail to deal with the switching delay $\Delta\tau$. Now, let us focus on the controllers in (7.23) designed via (7.7). They are active during the interval $[t_{i,0},\, t_{i+1,0})$, which implies that the controllers are designed considering the switching delays based on (7.7). Therefore, we conclude that the proposed Lyapunov function (7.7) reflects the key feature of switched linear systems with switching delays (as explained more technically in *Remark 7.1*).

7.6 Concluding remarks

In this chapter, a novel Lyapunov function for switched linear systems with switching delays has been proposed. Different from the classical Lyapunov function introduced by Branicky, this Lyapunov function is continuous at the switching instants and discontinuous when the system modes and controller modes are matched, which is consistent with the essence of switched systems with switching delays. A new stability condition via dwell time switching has been introduced to guarantee asymptotic stability in the noiseless case. Moreover, the proposed Lyapunov function can be used to guarantee a non-weighted $\mathcal{L}_2$ gain for switched linear systems with switching delays. Future work will focus on applying the proposed stability analysis to adaptive control of switched systems with switching delays and non-polytopic parametric uncertainties.

Chapter 8

Conclusions and Recommendations

In this thesis, we have introduced new adaptive control techniques for switched linear systems that fill the theoretical gap of asymptotic stability between adaptive control of non-switched linear systems and adaptive control of switched linear systems. In addition, robust and adaptive stabilization of switched linear systems with time delays have been studied. In this chapter, the main results of this thesis and some recommendations for future research are presented.

8.1 Conclusions

The main results of this thesis are summarized as follows:

- **Adaptive tracking control of uncertain switched systems using extended dwell time and average dwell time**

  We have extended the results in [91, 117] using extended dwell time and average dwell time switching, namely, mode-dependent dwell time, mode-mode-dependent dwell time, and mode-dependent average dwell time switching. These switching laws give rise to less conservative switching signals. Global uniform ultimate boundedness of switched linear system via the adaptive control schemes has been shown. Furthermore, an upper bound and an ultimate bound characterizing the transient and steady-state performance of the tracking error are introduced.

- **Adaptive asymptotic tracking of uncertain switched systems**

  A new adaptive law for uncertain switched linear systems has been proposed. In particular, a novel piecewise adaptive law has been proposed that gets rid of parameter projection and of the *a priori* knowledge of upper and lower bounds for the parameters when the switched system is not subject to disturbances. The proposed adaptive law and switching law based on dwell time guarantee that the tracking error converges to zero asymptotically. Furthermore, if the reference is persistently exciting, asymptotic stability becomes exponential and the parameter estimates of the state feedback controller converge to the real parameters.

- **Robust adaptive tracking of switched linear systems**

We have modified the adaptive law for switched linear systems exploiting the ideas of a parameter projection and a leakage method to preserve robustness of the closed-loop system in the presence of disturbances. In particular, global uniform ultimate boundedness has been achieved, and the ultimate bounds of the tracking error for both cases have been given.

- **Adaptive stabilization of switched systems with time-varying delays**

  We have developed an adaptive design for uncertain switched linear systems that can handle impulses in states and discontinuous time-varying delays at the switching instants. Using a new Lyapunov function that is non-increasing at the switching instants, we have designed an adaptive control law with a piecewise time-varying gain. Global uniform ultimate boundedness of the adaptive closed-loop system is guaranteed. The main contribution is that the impulsive behavior of both the states and the time-varying delay is addressed and solved in the presence of uncertainty.

- **Robust stability and stabilization of switched systems with switching delays**

  We have introduced a Lyapunov function for switched linear systems with switching delays: this Lyapunov function is continuous at switching instants and possibly discontinuous at the instant when the controller and the system mode is matched. Based on this Lyapunov function, a robust controller has been developed to achieve a non-weighted $\mathcal{L}_2$ gain for switched linear systems with switching delays.

8.2 Recommendations for future research

In this section, some potential topics based on the research of this thesis are introduced.

The first family of future research topics stems from the results of Chapters 3–5, which is proposed as follows:

- **Adaptive asymptotic tracking control with average dwell time switching**

  Asymptotic stability of the tracking error has been guaranteed in Chapter 4 based on dwell time switching signals. To reduce the conservativeness of dwell time switching, it is of theoretical and practical interest to investigate adaptive asymptotic tracking control with average dwell time switching. The difficulty consists in developing a stability condition with average dwell time such that the Lyapunov function is non-increasing at the switching instants.

- **Improving transient performance of the tracking error**

  As shown in Chapters 3–5, at switching instants, the tracking error may be subject to high-frequency oscillations caused by parameters changes. This could violate the physical limitations of the systems or even lead to failures of the controllers. In this regard, it is worth investigating how to decrease the oscillations. One possible solution is to incorporate a low-pass filter in the control loop as in $\mathscr{L}_1$-adaptive control [16, 41] where fast adaptation is allowed. The difficulty consists in guaranteeing asymptotic stability.

- **Addressing large modeling uncertainty**

When the modeling uncertainty of each subsystem is so large that a single adaptive controller for each subsystem cannot guarantee satisfactory tracking performance, a new adaptation mechanism is required. Adaptive control using multiple models [49, 71, 72] would be an efficient method to address large modeling uncertainty, where the multiple models essentially formulate a switched system. The switching between the switched system depends on the regions of the tracking error, which turns out to be a state-dependent switching. The difficulty is to incorporate two different kinds of switching signals, i.e., time-driven and state-dependent switching signals.

- **Adaptive output-feedback control of switched systems**

  The adaptive controllers proposed in Chapters 3–5 depend on the state signals. However, it is clear that in many cases the measurements of the states are not economical or even technically impossible. In this regard, designing an adaptive controller based on output signals is relevant and not trivial. The difficulty is to design a switching law and an output-feedback adaptive law based on the switched linear system without the knowledge of the system matrices.

The second family of future research topics comes from the results of Chapter 6, which is proposed as follows:

- **Adaptive tracking control of switched time-delay systems**

  Since adaptive regulation of switched time-delay systems has been addressed in Chapter 6, it is relevant to consider adaptive tracking when a family of reference models is involved. In the setting of adaptive tracking, the Lyapunov function (6.15) is not applicable and thus a Lyapunov function with the summation of parameter estimation errors should be used because a family of parameter estimates are needed instead of one for the case of (6.15) . This will give rise to a new family of adaptive laws different from that of Chapter 6 to guarantee stability.

- **New adaptive design via differential Riccati equation**

  The adaptive controllers (6.11) have been developed by solving a family of algebraic Riccati equations. To guarantee that the Lyapunov function (6.15) with a family of constant symmetric positive definite matrices P_p is decreasing at the switching instants, the time-varying coefficient should be carefully designed. One potential method to remove the time-varying coefficient is resorting to solutions to a family of differential Riccati equations, i.e., time-varying symmetric positive definite matrices $P_p(\cdot)$. By making using of $P_p(\cdot)$, we can impose the decreasing condition for the resulting Lyapunov function $V(t) = x^T(t)P_{\sigma(t)}(t)x(t) + \frac{1}{2\gamma}\tilde{\theta}^2(t),\ t \geq 0$ at the switching instants.

The third family of future research topics comes from the results of Chapter 7, which is listed as follows:

- **Adaptive control of switched systems with switching delays**

  In Chapter 7, a Lyapunov function that is continuous at the switching instants and discontinuous at the matching instants is proposed for switched systems with time delays. Since the Lyapunov function with parameter estimate errors does not admit rates of the exponential decreasing during the matched intervals and of the exponential increasing during the unmatched intervals, a new adaptive law different from that of Chapter 4 is needed to guarantee asymptotic stability.

- **Addressing sample-data switched systems with switching delays**

  When the control signals are transmitted through a communication network, quantized signals are used to save bandwidth. The method proposed in Chapter 7 can be extended to design a dynamical quantizer [15, 59] that simplifies the design of the dynamic quantizer for switched linear systems in [106] by increasing the adjustable parameter μ during the zooming-out stage or unmatched intervals and decreasing μ during the zooming-in stage or matched intervals.

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Summary

Control of Switched Linear Systems: Adaptation and Robustness

As a special class of hybrid systems, switched systems have attracted a lot of attention in the last decade due to theoretical and practical interests. When controlling switched systems, a ubiquitous problem is the presence of large parametric uncertainties and external disturbances. However, the state of the art on adaptive and robust control of switched linear systems is not satisfactory and due to the existence of theoretical gaps between adaptive and robust control for switched linear systems and non-switched linear systems. To this end, this thesis has been successfully closed some theoretical gaps, which is divided into two parts.

In the first part of this thesis, to start with, we have extended the state-of-the-art results using extended notions of dwell time and of average dwell time: mode-dependent dwell time and mode-dependent average dwell time, respectively. This gives rise to less conservative switching signals. To address the cases in which the next subsystem to be switched on is known, we propose a new time-dependent switching scheme: mode-mode-dependent dwell time, which not only exploits the information of the current subsystem, but also of the next subsystem. Subsequently, an adaptive law for uncertain switched linear systems has been introduced, which fills the theoretical gaps between adaptive control of non-switched linear systems and of switched linear systems. The proposed adaptive law and switching law based on dwell time guarantee asymptotic convergence of the tracking error to zero and, with a persistent exciting reference input, convergence of parameter estimates to nominal parameters asymptotically. To conclude the first part of this thesis, the adaptive law for switched linear systems has been modified using the ideas of parameter projection and leakage method, depending on the available *a priori* information: when the bounds of uncertain parameters are known, parameter projection is adopted; otherwise, the leakage method is used. The resulting adaptive closed-loop system is shown to be global uniform ultimate bounded in the presence of external disturbances.

In the second part of this thesis, adaptive and robust stabilization of switched linear systems have been investigated. Based on the stability conditions, adaptive stabilization of uncertain asynchronously switched systems is studied. Furthermore, in the presence of discontinuous time-varying delays, neither Krasovskii nor Razumikhin techniques can be successfully applied to adaptive stabilization of uncertain switched time-delay systems. A new adaptive control scheme for switched time-delay systems is developed that can handle impulsive behavior in states and time-varying delays with discontinuities. At the core of the proposed scheme is a Lyapunov function with a dynamically time-varying coefficient, which allows the Lyapunov function to be non-increasing at the switching instants. The control scheme substantially enlarges the class of uncertain switched systems for which the adaptive stabilization problem can be solved. Furthermore, in the presence of switching delays between a mode change and activation of its corresponding controller, enhanced stability criteria are investigated, whose novelty consists in continuity of the Lyapunov function at the switching instants and discontinuity when the system modes and controller modes are matched. The proposed Lyapunov function can be used to guarantee a finite non-weighted L_2 gain for asynchronously switched systems, for which methods proposed in literature are inconclusive.

Shuai Yuan

Samenvatting

Regeling van geschakelde lineaire systemen: adaptatie en robuustheid

Als een speciale klasse van hybride systemen hebben geschakelde systemen het afgelopen decennium veel aandacht getrokken vanwege theoretische en praktische interesses. Bij het regelen van geschakelde systemen is een alomtegenwoordig probleem de aanwezigheid van grote parametrische onzekerheden en externe storingen. De stand van de techniek met betrekking tot adaptieve en robuuste regeling van geschakelde lineaire systemen is echter niet toereikend en vanwege het bestaan van theoretische openingen tussen adaptieve en robuuste regeling voor geschakelde lineaire systemen en niet-geschakelde lineaire systemen. Hiertoe heeft dit proefschrift enkele theoretische hiaten met succes afgesloten,verdeeld in twee delen.

In het eerste deel van dit proefschrift hebben we om te beginnen de state-of-the-art resultaten uitgebreid met behulp van uitgebreide noties van dwell-tijd en van gemiddelde dwell-tijd: modus-afhankelijke dwell-tijd en modus-afhankelijke gemiddelde dwell-tijd, respectievelijk. Dit geeft aanleiding tot minder conservatieve schakelsignalen. Om de gevallen aan te geven waarin het volgende subsysteem moet worden ingeschakeld, stellen we een nieuw tijdsafhankelijk schakelschema voor: mode-modus-afhankelijke dwell-tijd, die niet alleen de informatie van het huidige subsysteem, maar ook van het volgende subsysteem gebruikt. Vervolgens is een adaptieve wet voor onzekere geschakelde lineaire systemen geïntroduceerd, die de theoretische kloof opvult tussen adaptieve controle van niet-geschakelde lineaire systemen en van geschakelde lineaire systemen. De voorgestelde adaptieve wet en omschakelingswet op basis van dwell-tijd garanderen asymptotische convergentie van de volgfout tot nul en, met een persistent exciterende referentie-invoer, convergentie van parameterschattingen tot asymptotische parameters. Om het eerste deel van dit proefschrift af te sluiten, is de adaptieve wet voor geschakelde lineaire systemen aangepast met behulp van de ideeën van parameterprojectie en lekmethode, afhankelijk van de beschikbare *a priori* informatie: wanneer de grenzen van onzekere parameters bekend zijn, wordt parameterprojectie toegepast; anders wordt de lekmethode gebruikt. Het is aangetoond dat het resulterende adaptieve gesloten-lussysteem globaal uniform ultiem begrensd is in de aanwezigheid van externe verstoringen.

In het tweede deel van dit proefschrift is adaptieve en robuuste stabilisatie van geschakelde lineaire systemen onderzocht. Op basis van de stabiliteitsvoorwaarden wordt adaptieve stabilisatie van onzekere, asynchroon geschakelde systemen bestudeerd. Bovendien kunnen, in aanwezigheid van discontinue, in de tijdvariant vertragingen, noch Krasovskii noch Razumikhin-technieken met succes worden toegepast op adaptieve stabilisatie van onzekere geschakelde tijdvertragingssystemen. Een nieuw adaptief regelschema voor geschakelde tijdvertragingssystemen is ontwikkeld dat impulsief gedrag in toestanden en in de tijdvariant vertragingen met discontinuïteiten aankan. De kern van het voorgestelde schema is een Lyapunov-functie met een dynamisch in de tijdvariant coëfficiënt, waardoor de Lyapunov-functie niet toeneemt op de schakelmomenten. Het regelschema vergroot de klasse van onzekere geschakelde systemen waarvoor het adaptieve stabilisatieprobleem kan worden opgelost aanzienlijk. Verder worden, in de aanwezigheid van schakelvertragingen tussen een modusverandering en activering van de corresponderende regelaar, verbeterde stabiliteitscriteria onderzocht, waarvan de nieuwheid bestaat uit de continuïteit van de Lyapunov-functie op de schakelmomenten en discontinuïteit wanneer de systeemmodi en besturingsmodi overeenkomen. De voorgestelde Lyapunov-functie kan worden gebruikt om een eindige niet-gewogen L_2 winst voor asynchroon geschakelde systemen te garanderen, waarvoor in de literatuur voorgestelde methoden niet doorslaggevend zijn.

Shuai Yuan

List of Publications

Journal articles

1. S. Yuan, B. De Schutter, and S. Baldi, “Adaptive asymptotic tracking control of uncertain time-driven switched linear systems”, *IEEE Transactions on Automatic Control*, Vol. 62, Issue 11, pp. 5802–5807, 2017.

2. S. Yuan, L. Zhang, B. De Schutter, and S. Baldi, “A novel Lyapunov function for a non-weighted L_2 gain of asynchronously switched linear systems”, *Automatica*, Vol. 87, pp. 310-317, 2018.

3. S. Yuan, B. De Schutter, and S. Baldi, “Robust adaptive tracking control of uncertain slowly switched linear systems”, *Nonlinear Analysis: Hybrid Systems*, Vol. 27, pp. 1-12, 2018.

4. S. Yuan, F. Zhang, and S. Baldi, “Adaptive tracking of switched nonlinear systems with prescribed performance using a reference-dependent reparametrization approach”, *International Journal of Control*, accepted, 2018.

5. Y. A. Harfouch, S. Yuan, and S. Baldi, “An adaptive switched control approach to heterogeneous platooning with inter-vehicle communication losses”, *IEEE Transactions on Control of Network Systems*, accepted, 2018.

6. N. Moustakis, S. Yuan, and S. Baldi, “An adaptive design for quantized feedback control of uncertain switched linear systems ”, *International Journal of Adaptive Control and Signal Processing*, accepted, 2018.

7. S. Baldi, S. Yuan, P. Endel, and O. Holub, “Dual estimation: Constructing building energy models from data sampled at low rate”, *Applied Energy*, Vol. 169, pp. 81-92, 2016.

8. S. Korkas, S. Baldi, S. Yuan, and E. B. Kosmatopoulos, “An adaptive learning-based approach for nearly-optimal dynamic charging of electric vehicle fleets”, *IEEE Transactions on Intelligent Transportation Systems*, accepted, 2018.

9. S. Yuan, L. Zhang, Ondrej Holub, and S. Baldi, “Switched Adaptive Control of Air Handling Units with Discrete and Saturated Actuators”, *IEEE Control Systems Letters*, accepted, 2018.

International referred journals under review

1. S. Yuan, L. Zhang, and S. Baldi, “Adaptive stabilization of impulsive switched linear time-delay systems: a piecewise dynamic gain approach”, submitted to *Automatica*.

Conference proceedings

1. S. Yuan, B. De Schutter, and S. Baldi, “On robust adaptive control of switched linear systems”, *the 13th IEEE International Conference on Control and Automation*, July 3rd-6th, Ohrid, Macedonia, pp. 753-758, 2017.

2. S. Yuan and S. Baldi, “Stabilization of switched linear systems using quantized output feedback via dwell-time switching”, *the 13th IEEE International Conference on Control and Automation*, July 3rd-6th, Ohrid, Macedonia, pp. 236-241, 2017.

3. Y. A. Harfouch, S. Yuan, and S. Baldi, “An adaptive approach to cooperative longitudinal platooning of heterogeneous vehicles with communication losses”, *the 20th IFAC World Congress*, July 9th-14th, Toulouse, France, pp. 1382-1387, 2017.

4. S. Yuan, B. De Schutter, and S. Baldi, “Adaptive tracking control of switched linear systems using mode-dependent average dwell time”, *the 8th International Conference on Information Science and Technology*, June 30-July 6th, Cordoba, Granada, and Seville, Spain, 2018.

5. S. Yuan, F. Zhang, and S. Baldi, “Adaptive stabilization of nonlinearly parametrized switched systems with unknown time-varying delays”, *the 8th International Conference on Information Science and Technology*, June 30-July 6th, Cordoba, Granada, and Seville, Spain, 2018.

6. N. Moustakis, S. Yuan, and S. Baldi, “An adaptive approach to zooming-based control for uncertain systems with input quantization”, *the 16th European Control Conference*, June 12th-15th, Limassol, Cyprus, 2018.

Curriculum Vitae

Shuai Yuan was born in October 1988, Anxiang, Hunan Province, China. He received the B.Sc. and M.Sc. degree in Mechanical Science and Engineering from Harbin Institute of Technology, Huazhong University of Science and Technology, China, in 2011 and 2014 respectively.

In September 2014, he was sponsored by the Chinese Scholarship Council (CSC) to become a Ph.D candidate at Delft Center for Systems and Control, Delft University of Technology. In his Ph.D project, he worked on adaptive and robust control of switched systems, under the supervision of dr. ir. Simone Baldi and Prof. dr. ir. Bart De Schutter. From 1st July to 1st September in 2016, he was a visiting researcher in the Department of Building Automation, Honeywell Prague Laboratory. In addition, from 1st March to 31st May in 2018, he was a visiting student in Imperial College London hosted by Prof. Thomas Parisini.